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THE

AUTHOR'S PREFACE.

THE powers of the mind, like those of the body, are increased by frequent exertion ; application and industry supply the place of genius and invention ; and even the creative faculty itself may be strengthened and improved by use and perseverance. Uncultivated nature is uniformly rude and imbecile, since by imitation alone we at first acquire knowledge, and the means of extending its bounds. A just and perfect acquaintance with the simple elements of science, is a necessary step towards our future progress and advancement ; and this, assisted by laborious investigation and habitual inquiry, will constantly lead to eminence and perfection.

Books of rudiments, therefore, concisely written, well digested, and methodically arranged, are treasures of inestimable value ; and too many attempts cannot be made to render them perfect and complete. When the first principles of any art or science are firmly fixed in the mind, their application soon becomes easy, pleasant, and obvious ; the understanding is delighted and enlarged ; it conceives clearly, reasons distinctly, and forms just and satisfactory conclusions. But, on the contrary, when the mind, instead of reposing on the stability of truth and received principles, is wandering in doubt and uncertainty, our ideas will necessarily be confused and obscure ; and every step we take must be attended with fresh difficulties and endless perplexity.

That the grounds or fundamental parts of every science are unentertaining, is a complaint universally made, and a truth not to be denied ; but then, what is obtained with difficulty is usually remembered with ease; and what is purchased with pain is generally possessed with pleasure. The seeds of knowledge are sown in every soil, but it is by proper culture alone that they are cherished and brought to maturity. A few years of early and assiduous application never fail to procure us the reward of our industry; and who that knows the pleasures and advantages which the sciences afford, would think his time misspent or his labours useless? Riches and honours are the gifts of fortune, casually bestowed, or hereditarily received, and are frequently abused by their possessors; but the superiority of wisdom and knowledge is a pre-eminence of merit, which originates with the man, and is the noblest of all distinctions.

Nature, bountiful and wise in all things, has provided us with an infinite variety of scenes, both for our instruction and entertainment; and, like a kind and indulgent parent, admits all her children to an equal participation of her blessings. But, as the modes, situations, and circumstances of life are various, so accident, habit, or education has each its predominating influence, and gives to every mind its particular bias. Where examples of excellence are wanting, the attempts to attain it are few; but eminence excites attention, and produces imitation. To raise the curiosity, and to awaken the dormant powers of young minds, we have only to point out to them a valuable acquisition, and the means of obtaining it. The active principles are immediately put in motion, and the certainty of the conquest is ensured from a determination to conquer.

But, of all the sciences which serve to call forth this spirit of enterprise and inquiry, the Mathematical are the most eminently useful. By an early attachment to these elegant and sublime studies, we acquire a habit of reasoning, and an elevation of thought, which fix the mind, and prepare it for every other pursuit. From a few simple axioms, and evident principles, we proceed gradually to the most general propositions, and remote analogies; deducing one truth from another, in a chain of argument well connected and logically pursued; which brings us at last, in the most satisfactory manner, to the conclusion, and serves as a general direction in all our inquiries after truth.

And it is not only in this respect that mathematical learning is so highly valuable; it is, likewise, equally estimable for its practical utility. Almost all the works of art, and devices of men, have a dependence upon its principles, and are indebted to it for their origin and perfection. The cultivation of these admirable sciences is, therefore, an object of the highest importance, and ought to be considered as a principal part of every liberal and well-regulated plan of education. They are the guide of our youth, the perfection of our reason, and the foundation of every great and noble undertaking.

From these considerations, I have been induced to compose an introductory course of mathematical science; and, from the kind encouragement which I have hitherto received, I am not without hopes of a continuance of the same candour and approbation. Considerable practice as a teacher, and long attention to the difficulties and obstructions which retard the progress of learners, have enabled me to accommodate myself the more easily to their capacities and understandings. And as an earnest desire of promoting and diffusing useful knowledge is the chief motive for this undertaking, so every exertion has been made to render it as useful and complete as possible.

The subject of the present performance is ALGEBRA; which is one of the most important branches of those sciences, and may be justly considered as the key to all the rest. Geometry delights us by the simplicity of its principles, and the elegance of its demonstrations: Arithmetic is confined in its object, and partial in its application; but Algebra is general and comprehensive, and may be applied with success in all cases where truth is to be obtained, and proper data can be established.

To trace this science to its origin, and to point out the various alterations and improvements it has undergone in its progress, would far exceed the limits of a preface. It will be sufficient to observe, that the invention is of the highest antiquity, and has challenged the praise and admiration of all ages. Diophantus appears to have been the first, among the ancients, who applied it to the solution of indeterminate or unlimited problems; but it is to the moderns that we are principally indebted for the most curious refinements of the art, and its great and extensive usefulness in every abstruse and difficult enquiry. NEWTON, MACLAURIN, SANDERSON, SIMPSON, and EMERSON, among our own countrymen; and EULER, LAGRANGE, and LACROIX, on the continent, are those who have particularly excelled in this respect; and it is to their works that I would refer the young student, as patterns of elegance and perfection.

The following compendium is formed entirely upon the model of those writers, and is intended as a useful introduction to them. Almost every subject which belongs to pure Algebra, is concisely and distinctly treated of; and no labour has been spared to make the whole as easy and intelligible as possible. A great number of elementary books have already been written upon this subject; but all which I have yet seen appear to me to be extremely defective: besides being totally unfit for the purpose of teaching, they are generally calculated to vitiate the taste and mislead the judgment. A tedious method prevails through the whole, so that the beauty of the science is generally destroyed by the inelegant manner in which it is treated; and the learner, when he is afterwards introduced to some of our best writers, is obliged to discard much of that which he has been at so much pains in acquiring.

There is a certain elegance in the sciences, as well as in every branch of polite literature, which is only to be obtained from the best authors, and a judicious use of their instructions. To direct the student in his choice of books, and to prepare him properly for the advantages he may receive from them, is, therefore, the business of every writer who engages in the humble but useful task of a preliminary tutor. This information I have been careful to give, in every part of the present performance, where it appeared to be in the least necessary; and though the nature and confined limits of my plan admitted not of diffuse observations, or a formal enumeration of particulars, it is presumed nothing of real use and importance has been omitted.

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METHOD OF TEACHING RECOMMENDED.

1. To enjoin the student to read attentively the definitions, so as to understand their import completely, and ultimately to commit those essential accurately to memory.

2. To study the rules carefully, and to comprehend fully the principles involved in their structure and composition.

3. To follow out, independently, the operations employed in all those questions whose solutions are performed at length, to prepare the mind fully to comprehend the methods required in the Exercises afterwards prescribed.

4. If the pupil is very young, or just commencing his studies, attention may be directed to the solution of a moderate number of the more simple Examples, which, for this purpose, have been placed first in order.

5. Having proceeded in this manner as far as may be thought requisite, he may return to a revisal of these; while, at the same time, he may, at the discretion of his Instructor, undertake the solution of the whole, or such a portion of the general Exercises, as may be considered advisable, which are particularly designed for more advanced students.

6. In this manner the Treatise may become either a simple and popular work, or as complete as its limits will permit, so far as the experience of the Editors could determine.

7. The Key, containing the solutions of all the unresolved Exercises, with a few Additional Remarks and Illustrations, will enable teachers and private students to use this Treatise advantageously, and without difficulty.

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ELEMENTS OF ALGEBRA.

PRELIMINARY REMARKS.

Algebra is that branch of science which is employed to perform calculations by means of the letters of the alphabet, combined with certain signs and symbols.*

The letters of the alphabet are used to denote quantities or magnitudes of any kind, while the signs indicate the operations to be performed upon them.

By this combination, algebra may be very advantageously employed in the most intricate investigations and profound speculations regarding the material world, to which the human mind can be applied.

Quantity or *magnitude* is anything which will admit of increase or decrease, or that is capable of being measured or numbered; such as *lines*, *space*, *volume*, *time*, *motion*, *weight*, *force*, &c.

Quantities are divided into two kinds or classes; distinguished by the terms *known* and *unknown*, or by the terms *given* and *required.*

Known or *given* quantities are usually represented, in the science of algebra, by the early letters of the alphabet, as a, b, c, d, &c., while those that are *unknown* or *required*, are denoted by the final letters, as z, y, x, v, &c.

Since letters, in this science, have no *numerical* significations in themselves, they are subject to no ambiguity, each retaining, throughout the whole of any investigation, the signification which it was defined to have at the commencement.

---

* The following is a neat and concise definition:—Algebra is a general method of reasoning upon magnitude and quantity by means of symbolical characters.

The object of the science of algebra is, by solution, either the investigation of problems, or the demonstration of theorems.

The intention of the *algebraical problem* is the discovering of certain unknown quantities, by means of a relation between these and known quantities, indicated by the conditions of the question; or it is the discovery of formulæ for the solution of questions.

The object of the *algebraical theorem* is the demonstration of certain properties and relations which exist in the combination of known and given quantities.

DEFINITIONS AND NOTATION.

The following are the principal symbols employed to indicate the operations in algebra, and the relations subsisting between different quantities:—

1. The sign + (*plus*) denotes that the quantity which it *precedes* is to be *added*, and all quantities before which + stands, are called *positive quantities.* Thus $3 + 6$ signifies that 6 is to be added to 3; and $a + b$, which is read *a plus b*, signifies that the quantity denoted by b is to be added to that denoted by a.

2. The sign − (*minus*) denotes that the quantity which it *precedes* is to be subtracted, and all quantities before which − stands, are called *negative quantities.* Thus, $6 - 4$ signifies 6 *minus* 4, or 6 diminished by 4, and $a - b$ indicates that the quantity denoted by b is to be subtracted from that designated by a.

3. The sign ± (*plus* or *minus*) comprehends both the former signs, and denotes that the quantity before which it stands is to be added when + is used, and subtracted when − is employed.

If neither + nor − precede a quantity, + is always understood, and the quantity is reckoned positive: thus a means $+ a$.

4. The sign $\backsim$ (*difference*), signifies the difference of the quantities between which it is placed, when it is not known which is the greater.

5. The sign = (*equal*) denotes that the simple quantity, or the aggregate of the quantities which precede it, is equal to the simple quantity, or to the aggregate of the quantities which succeed it. Thus, $2 + 3 - 1 = 10 - 2 - 4$, or $a = b + c$, signifies that a *is equal to* the sum of b and c.

6. The sign × (*into*) denotes that the quantities between which it

is placed are to be multiplied together. Thus, 4×8 signifies 4 *into* 8, or 4 *multiplied* by 8; $a \times b \times c$, indicates the continued product of the quantities represented by a, b, c, called *factors*, and so on for any number of them.

The process of multiplication is also frequently indicated by placing a dot or point between the successive factors; thus $a\,.\,b\,.\,c\,.\,d$ signifies the same as $a \times b \times c \times d$; and when quantities are represented by letters, their continued product is indicated simply by writing the letters in succession, without interposing either the sign × or . between them. Thus, $abcd$ has the same signification as $a \times b \times c \times d$, or $a\,.\,b\,.\,c\,.\,d$, and is more concise.

The multiplication of numbers, however, can only be expressed by the point in cases where no ambiguity can arise from its use; thus, 1. 2. 3. 4 may represent the continued product of 1, 2, 3, 4; and $\frac{1}{2}.\,\frac{3}{4}.\,\frac{5}{6}$ the continued product of the fractions $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$; while 7.8 may be mistaken for 7·8, that is, for seven and eight-tenths, and 78 for the number *seventy-eight.*

7. The sign ÷ (*by*) denotes that the quantity which *precedes* it is to be divided by the quantity which succeeds it. Thus $a \div b$ signifies that a is to be divided by b, and read a by b, or a divided by b.

The operation of division is, however, more usually indicated by writing the dividend above the line, and the divisor below it, in the places of the two points, (which indicate symbolically their position), in the manner of a fraction: thus, $16 \div 8$ and $a \div b$ are usually written $\frac{16}{8}$ and $\frac{a}{b}$.

8. The sign $<$ (*inequality*) denotes that one of the quantities between which it is placed is *greater* than the other, the opening of the sign being turned towards the greater quantity. Thus, $a < b$ signifies that a is less than b, and $a > b$ denotes that a is *greater* than b.

9. The sign : (*is to*), (somewhat analogous to the sign of *division*), denotes that the quantity which precedes it has the same relation to that which succeeds it, that the former *divided* by the latter has to 1. Thus, 4 : 2 denotes that 4 *is to* 2, or that 4 has the same relation to 2 that $\frac{4}{2}$ has to 1; for 4 is double of 2, and $\frac{4}{2}$, or 2, is also double of 1. Hence, since the former number divided by the latter is always compared with unity, the value of the fraction $\frac{4}{2}$, or 2, is termed the relation or *ratio* of these two numbers. If any two other numbers be taken, so that the number arising by dividing the one by the other

has the same relation to unity, then these latter numbers have the same relation or *ratio* to each other. Thus, since $\frac{8}{4}$ or 2 has the same relation to 1 that $\frac{4}{2}$ has to 1; then 4 is said to have to 2 the same relation or *ratio* that 8 has to 4.

The same is true of all quantities that can be numerically expressed.

10. The signs : :: : (*proportion*) denote that the four quantities between which they are placed, are each consecutive pair, equal ratios thereby constituting a proportion; thus $x:y::a:b$, signifying that x has to y the same ratio that a has to b, commonly read, *x is to y as a is to b*, or as x is to y, so is a to b.

11. The *coefficient* of a quantity is the number or letter prefixed to it, and indicates how many times the quantity is to be taken, or by what quantity it is to be multiplied: thus, $5a$ signifies that the quantity a is to be taken five times, and consequently the number 5 is the coefficient of a. Also, in the expressions ax and $4abx$, the coefficients are respectively a and $4ab$, thus combining numbers with the early letters of the alphabet.

When no numeral coefficient is prefixed to a quantity, its coefficient is *always* 1 *understood:* thus a and x are the same as $1a$ and $1x$, though 1 is never written or expressed in practice.

12. A *power* of a quantity is the product arising from the multiplication of that quantity any given number of times by itself: thus $a \times a$ is the second power or square of a, $a \times a \times a$ is the third power or cube of a, and so on. Instead, however, of repeating the factor the required number of times, a small figure is placed on the right at its upper corner, and expresses the number of factors to be multiplied together. Thus, a^2 and a^3 denote the square and cube of a respectively, and x^n denotes the power of x when the same factor is extended to n terms.

13. A *root* of a quantity is that quantity which, when multiplied by itself a certain number of times, produces the proposed quantity: thus a is the *second* or *square* root of a^2, for $a \times a = a^2$; x is the *third* or *cube* root of x^3, for $x \times x \times x = x^3$; and y is the nth root of y^n, for $y \times y \times y \times \ldots .$ to n factors $= y^n$.

The sign $\sqrt{}$*, which is named the *radical sign*, placed before a quantity, indicates that some root is to be taken, and a small figure placed over the sign, or rather in the angle, shews the root to be extracted, except in the case of the square root, where the figure 2 is always understood.

---

* In the early and imperfect state of algebra, a contraction of the requisite word was frequently used—often the initial letter, whence this character is merely the antiquated form of the letter *r*, the initial letter of the word *root*.

Roots of quantities are frequently and generally more conveniently denoted by fractions, placed at the right upper corner of the quantity whose root is to be extracted, thus:—

The square root of a is indicated either by $\sqrt{a}$ or $a^{\frac{1}{2}}$
„ cube root „ „ $\sqrt[3]{a}$ or $a^{\frac{1}{3}}$
„ fourth root „ „ $\sqrt[4]{a}$ or $a^{\frac{1}{4}}$
„ nth root „ „ $\sqrt[n]{a}$ or $a^{\frac{1}{n}}$
„ $\frac{m}{n}$th root „ „ $\sqrt[n]{a^m}$ or $a^{\frac{m}{n}}$

This last meaning that the nth root of a, raised to the mth power, is *really*, or *understood* to be, extracted.

In the operations of algebra, the identity of these expressions must always be recollected.

14. The reciprocal of an integer quantity is unity divided by that quantity. Thus the reciprocals of a^2, x^3, y^5, $a^{\frac{1}{2}}$, $x^{\frac{1}{n}}$, are respectively $\frac{1}{a^2}, \frac{1}{x^3}, \frac{1}{y^5}, \frac{1}{a^{\frac{1}{2}}}, \frac{1}{x^{\frac{1}{n}}}$; but these reciprocals are frequently represented in the more commodious form a^{-2}, x^{-3}, y^{-5}, $a^{-\frac{1}{2}}$, $x^{-\frac{1}{n}}$. The reciprocal of a general fraction $\frac{a^2}{b^3}$ is $\frac{b^3}{a^2}$, by the inversion of its terms.

15. The *exponent* or *index* of a quantity is the small figure or fraction employed to denote the power to which it is to be raised, or the root which is to be extracted, as well as that employed to denote the reciprocal of any power or root. Thus the exponents or indices of a^2, $x^{\frac{1}{3}}$, y^{-2}, are respectively 2, $\frac{1}{3}$, and -2, the latter being a *negative* exponent, indicating the reciprocal of y^2. When no exponent is expressed, unity is understood; thus x is the same as x^1, and this must be always recollected.

16. The *terms* of an expression are those parts of it connected by the signs + or −, and the expression itself is either *simple* or *compound.*

A *simple* quantity consists of a single term, as $4ay$.

A *compound* quantity consists of two or more terms, as $a-3b+4x$.

17. *Brackets,** ( ), { }, [ ], are employed to denote that all the quantities within them are, in their aggregate, to be considered as forming

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* A *vinculum* or line over the quantities was formerly much used for the same purpose; but being less convenient, is now seldom employed. The vinculum may be drawn either horizontally or vertically.

one quantity only. The effect of the use of these brackets must be carefully attended to. Thus $x-(y-z)$ is *not* the same as $x-y-z$, because in the former expression $y-z$, or the excess of y above z, is the compound quantity which is to be subtracted, and in the latter both y and z are to be subtracted. This operation in algebra is effected by changing the signs of $y-z$ in the first quantity, while those of the latter remain unchanged, as will be explained in the rule for the subtraction of algebraic quantities.

18. *Like* quantities are those which contain the *same* letters or the *same* combination of letters, as a, $3a$; $2x^2y$, $3x^2y$; $4x^{\frac{1}{2}}$, $7\sqrt{x}$; &c.

Unlike quantities are those which contain *different* letters, or *different combinations* of letters, as a, b, $3x^2y$, $4xy^2$, $5xy^{\frac{1}{2}}$, $7x^{\frac{1}{2}}y^{\frac{2}{3}}$, &c.

19. A *monomial* quantity consists of one term only, as a, bx, cxy, &c.

A *binomial* quantity comprises two terms, as $x-y$, $ab+cd$, &c.*

A *trinomial* involves three terms, as $a+bx+cy$, &c.

A *multinomial* or *polynomial* quantity is composed of *several* simple terms, when their number is not precisely defined.

20. The sign $\therefore$ is frequently employed chiefly in this country for *therefore* or *consequently*, and the sign $\because$ for *since* or *because.*

21. The sign ∞ (*infinity*) denotes that the quantity standing before it is of an unlimited value.

22. The sign $\propto$ (*variation*) signifies that the quantity which precedes it varies as that which follows.

Thus $\text{A} \propto \text{B}$, A varies *directly* as B, $\text{A} \propto \frac{1}{\text{B}}$, A varies *inversely* or *reciprocally* as B; $\text{A} \propto \text{B} \times \text{C}$, that is, A varies *conjointly* as B and C; and $\text{A} \propto \frac{\text{B}}{\text{C}}$, or A varies *directly* as B, and *inversely* as C.

23. A *function* of one or more quantities is an expression into which those quantities enter either alone or connected with other quantities, as $a+x$, $bx+cx^2$, $(a^2-x^2)^{\frac{1}{2}}=y$.

Functions are divided into classes as algebraic functions, transcendental functions, trigonometrical functions, such as sines, tangents, &c.

For the purpose of illustrating the preceding definitions, the numerical valuations of some algebraical expressions are subjoined.

Find the numerical values of the following algebraical expressions:—

If $a=1$, $b=3$, $c=2$, $d=0$, and $e=5$.

1. $a+b-2c+5d+2e=1+3-4+0+10=10$.

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* The binomial $x-y$ is sometimes called a *residual* quantity.

2. $ab-cd+ac+ae-bd=3-0+2+5-0=10.$

3. $(a+b)(c+d)+(a+c)(b-d)=4\times 2+3\times 3=8+9=17.$

4. $a^4-4a^3c+6a^2c^2-4ac^3+c^4=1-8+24-32+16=1.$

5. $\dfrac{a+b}{c-d}+\dfrac{a+c}{e-c}-\dfrac{b+d}{a+e}=\dfrac{4}{2}+\dfrac{3}{3}-\dfrac{3}{6}=3-\frac{1}{2}=2\frac{1}{2}.$

6. $\sqrt{a}+\sqrt{3b}-\sqrt{2c}+2\sqrt{4a}=1+3-2+4=6.$

EXAMPLES FOR EXERCISE.

Find the values of the following expressions, if

$$a=2,\ b=1,\ c=4,\ d=6,\ e=9.$$

1. $(a+b+c)(a+c+e)$? Ans. 105.

2. $\dfrac{ab+ac-bc}{2a+3b}+\dfrac{ac-bd+ae}{3c+d}-\dfrac{2ab+3ad-ce}{e-(b+c)}$? Ans. $\dfrac{61}{63}$.

3. $\{d-a+(b+c)^2\}.\{(c-b)^2-(e-d)\}.\{(a+b+c+d)^2-3de\}$? Ans. 1218.

4. $2\sqrt{(a-b)}+3\sqrt{c}-\sqrt{ce}+\sqrt[3]{(e-b)}-\sqrt{(a+b+c+d-e)}$? Ans. 2.

5. $\sqrt[3]{\{ac-(b+d)\}}+\sqrt[3]{\{(2d+3b)^2-2e(a+e)\}}-\sqrt[3]{abc}$? Ans. 2.

6. Shew that if $a=5$, $b=4$, and $c=3$,
$(a+b+c)\ (a+b-c)\ (a+c-b)\ (b+c-a)=4a^2b^2-\{a^2+b^2-c^2\}^2.$

7. If $a=5$, $b=4$, $c=3$, $d=2$, shew that $4\,(a^2+b^2+c^2+d^2)=(a+b+c-d)^2+(a+b+d-c)^2+(a+c+d-b)^2+(b+c+d-a)^2.$

8. If any numerical value be given to a greater than to b and any to n, shew that $a^2-b^2=(a+b)(a-b).$
$$a^3+b^3=(a+b)(a^2-ab+b^2).$$
$$a^3-b^3=(a-b)(a^2+ab+b^2).$$
$$a^{2n}-b^{2n}=(a^n+b^n)(a^n-b^n).$$

The component factors of these formulæ should be recollected for their utility in future applications, particularly in fractions.

ADDITION.

1. ADDITION is the method of connecting quantities together by means of the signs prefixed to them, and incorporating such as are *like*, which alone can be *united*, into one sum.

2. *Unlike quantities* cannot be incorporated, and consequently must be written in succession, prefixing to each its proper sign.

3. It will often be found convenient to embrace within brackets the coefficients, prefixing or subjoining such a common factor as may be suitable.*

Thus, the sum of a and $-2x$ is $a-2x$, the sum of $4x-2y$ and $3a+2b$ is $4x-2y+3a+2b$; and the sum of ax^2+bx+c, and mx^2-nx-d is $ax^2+mx^2+bx-nx+c-d$, or $(a+m)x^2+(b-n)x+c-d$,

$$\begin{array}{r|r|r|} \text{or } a & x^2+b & x+c \\ m & -n & -d \end{array}$$

This last or third method is not frequently used. The addition of like quantities might therefore be properly divided into the two following cases:—

(1.) *Like* quantities with *like* signs.

(2.) *Like* quantities with *unlike* signs.

In compliance with common practice, we shall still retain another case.

(3.) *Unlike* quantities with *unlike* signs.†

CASE I.

When the quantities are like, and have like signs.

Rule.—Add the coefficients together, prefixing the common sign if necessary, and annexing the common letter or letters.

EXAMPLES.

(1.)	(2.)	(3.)	(4.)	(5.)
a	$2ax$	$-3ay$	$-2by^2$	$abcx$
$3a$	$5ax$	$-6ay$	$-6by^2$	$3abcx$
$5a$	$4ax$	$-ay$	$-by^2$	$2abcx$
$7a$	$7ax$	$-2ay$	$-8by^2$	$7abcx$
$12a$	$16ax$	$-7ay$	$-by^2$	$abcx$
$28a$	$34ax$	$-19ay$	$-18by^2$	$14abcx$

---

* The term Addition, which is generally used to denote this rule, is too scanty to express the nature of the operations that are to be performed in it; which are sometimes those of addition, and sometimes subtraction, according as the quantities are positive or negative. It should, therefore, be called by some name signifying incorporation, or striking a balance; in which case, the incongruity here mentioned would be removed.

† According to our remarks in article 2 of addition, the incorporation of unlike quantities cannot be effected in the usual sense, which would thus have reduced the cases strictly to *two*.

(6.)	(7.)	(8.)	(9.)
$2b + 3y$	$a - 2x^2$	$7x - 4y$	$2a + x^2$
$5b + 7y$	$a - 6x^2$	$x - 8y$	$3a + x^2$
$b + 2y$	$4a - x^2$	$3x - y$	$a + x^2$
$8b + y$	$3a - 5x^2$	$x - 3y$	$9a + x^2$
$4b + 4y$	$7a - x^2$	$4x - y$	$4a + x^2$

CASE II.

When the quantities are unlike, and have unlike signs.

Rule.—Add first the positive, and then the negative quantities, now subtract the less sum from the greater; prefix, if necessary, the sign of the greater sum to the remainder, and annex the common letter or letters.

EXAMPLES.

(1.)	(2.)	(3.)	(4.)	(5.)
$-3a$	$-2a^2$	$-2a\sqrt{x}$	$2a - 3x^2$	a^{-2}
$+7a$	$-3a^2$	$+7a\sqrt{x}$	$-7a + 6x^2$	$2a^{-2}$
$+8a$	$-8a^2$	$-9a\sqrt{x}$	$-3a + x^2$	$4a^{-2}$
$+a$	$+10a^2$	$+7a\sqrt{x}$	$+a - 3x^2$	$-a^{-2}$
$-a$	$+16a^2$	$-4a\sqrt{x}$	$+2a - x^2$	$-3a^{-2}$
$+12a$	$13a^2$	$-a\sqrt{x}$	$-5a$	$3a^{-2}$

(6.)	(7.)	(8.)
$2ay - 7$	$-px^2 + qy$	$-6a^2 + 2b + 6ax^2 + 5x^{\frac{1}{2}}$
$-ay + 8$	$+5px^2 - 2qy$	$+2a^2 - 3b - 2ax^2 - 6x^{\frac{1}{2}}$
$+2ay - 9$	$+7px^2 - 6qy$	$-5a^2 - 8b + 3ax^2 - 10x^{\frac{1}{2}}$
$-3ay - 11$	$-2px^2 + 5qy$	$+4a^2 - 2b - 7ax^2 + 3x^{\frac{1}{2}}$
$+12ay + 13$	$+9px^2 + 2qy$	$-3a^2 + 9b + ax^2 + 11x^{\frac{1}{2}}$

(9.)	(10.)	(11.)	(12.)
ax	$2ay^2$	$3ax - 2by$	$amx^2 + bny + rz$
bx	$-3by^2$	$5cx + 3dy$	$cpx^2 - dqy - brz$
cx	$+4ay^2$	$-2dx - 4ay$	$-qx^2 + py + arz$
dx	$-by^2$	$-4hx + 3ky$	$-nx^2 - my - crz$

EXERCISES.

Find the sums of the following quantities:—*

1. $5xy$, $4ax$, $-xy$, and $-4ax$. Ans. $4xy$.
2. $ax^{\frac{1}{2}}$, $-ax^2$, $+3ax^2$, and $-ax^{\frac{1}{2}}$. Ans. $2ax^2$.
3. $3x^2y$, $2xy^2$, $-7xy^2$, $-8x^2y$, and $2xy^2$. Ans. $-5x^2y-3xy^2$.
4. $2xy-2x^2$, $3x^2+xy$, x^2+xy, and $4x^2-3xy$. Ans. $6x^2+xy$.
5. $8a^2x^2-3ax$, $7ax-5xy$, $9xy-5ax$, and $2a^2x^2+xy$. Ans. $10a^2x^2-ax+5xy$.
6. x^3+3x^2-4x+2, $2x^3-5x^2+7x-8$, $3x^3+7x^2-5x+2$, x^3-5x^2-5x-1, and $4x^3+5x^2+2x+5$. Ans. $11x^3+5x^2-5x$.
7. $x+y+z-v$, $x+y-z+v$, $x-y+z+v$, and $-x+y+z+v$. Ans. $2(x+y+z+v)$.
8. $x^3-y^3+3x^2y-5xy^2$, $3x^3-4x^2y+3y^3-3xy^2$, $x^3+y^3+3x^2y$, $2x^3-4y^3-5xy^2$, $6x^2y+10xy^2$, and $-6x^3-7x^2y+4xy^2+2y^3$. Ans. $x^3+x^2y+xy^2+y^3$.
9. $\sqrt{}(a^2+b^2)-\sqrt{}(a^2-b^2)-5ab$, $2\sqrt{}(a^2+b^2)-3ab-5\sqrt{}(a^2-b^2)$, $10\sqrt{}(a^2-b^2)-12\sqrt{}(a^2+b^2)+7ab$, $8ab-2(a^2+b^2)^{\frac{1}{2}}-3(a^2-b^2)^{\frac{1}{2}}$, $ab+(a^2-b^2)^{\frac{1}{2}}+(a^2+b^2)^{\frac{1}{2}}$, and $12(a^2+b^2)^{\frac{1}{2}}-5(a^2-b^2)^{\frac{1}{2}}-4ab$. Ans. $2(a^2+b^2)^{\frac{1}{2}}-3(a^2-b^2)^{\frac{1}{2}}+4ab$.
10. Find the sum of $\frac{x+y}{2}$ and $\frac{x-y}{2}$. Ans. x.
11. Find the sum of $\frac{a^2+ab+b^2}{2}$ and $\frac{a^2-ab+b^2}{2}$. Ans. a^2+b^2.
12. Find the sum of x^2+xy+y^2, $mx^2+nxy+py^2$, and $qx^2+rxy+sy^2$. Ans. $(1+m+q)x^2+(1+n+r)xy+(1+p+s)y^2$.
13. Find the sum of $(a+b)x^2-(c+d)y^2$, $3(a+b)x^2+5(c+d)y^2$, and $6(a+b)x^2-4(c+d)y^2$. Ans. $10(a+b)x^2$.

CASE III.

When some of the quantities are like, and others unlike.

Rule.—Collect all the like quantities together, by taking their sums or differences, as in the foregoing cases, and set down those that are unlike, one after another, with their proper signs.

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* The *commas* shew generally the different lines of the example to be placed under each other, to obtain the aggregate or sum.

EXAMPLES.

(1.)

$$\begin{array}{c} ax^4 - bx^2 \\ cx^3 - dx \\ \hline ax^4 + cx^3 - bx^2 - dx \\ \text{or } (ax^3 + cx^2 - bx - d)\,x \end{array}$$

(2.)

$$\begin{array}{c} ex^{3m} - fx^n \\ gx^{2m} + hx^{\frac{1}{2}n} \\ \hline ex^{3m} + gx^{2m} - fx_n + hx^{\frac{1}{2}n} \\ \text{or } (ex^m + g)\,x^{2m} - (fx^{\frac{1}{2}n} - h)x^{\frac{1}{2}n} \end{array}$$

(3.)

$$\begin{array}{c} ax - bx \\ -cx + dx \\ gx - hx \\ \hline ax + dx + gx - bx - cx - hx, \\ \text{or } (a+d+g)\,x - (b+c+h)\,x \end{array}$$

(4.)

$$\begin{array}{r} + axy + \tfrac{1}{2}xy - 13 \\ - 2xy + cy - 7 \\ - 4cy - xy + 9 \\ \hline (a - 2\tfrac{1}{2})\,xy - 3cy - 11 \\ \text{or } (ax - 2\tfrac{1}{2}x - 3c)y - 11 \end{array}$$

(5.)

$$\begin{array}{c} ax^m + bx^n - cx^p + dx^q \\ -bx^m - dx^p + ax^n - ex^q \\ (e + f)\,x^n - (d + e)\,x^q + (b+c)\,x^p \\ (c - a)\,x^q - kx^p + (d + e)\,x^q \\ \hline (a-b)\,x^m + (a + b + e + f)\,x^n + (b-d-k)\,x^p - (a-c-d+e)x^q \end{array}$$

In collecting *coefficients* of aggregates in this manner, the following rules should be observed.

(1.) Collect the coefficients of the same power throughout the whole expression in *numerical* or *literal* order, as

$$1,\ 2,\ 3\ \ldots\ n,\ \text{or}$$
$$n\ \ldots\ 3,\ 2,\ 1.$$

(2.) In like manner, arrange the aggregates of coefficients in the order of the characters selected.

(3.) The second answer to the preceding questions may be omitted on a first perusal, and deferred till a knowledge of multiplication and division is obtained.

SUBTRACTION.

Subtraction is the method of finding the difference between any two quantities of the same kind.

Rule.—Conceive the sign or signs of the quantity or quantities to be subtracted to be changed, (+ into −, and − into +,) and proceed as in the several cases of addition.*

EXAMPLES.

(1.)

$$\begin{array}{r} 5a + 7b + 3c \\ 3a + 2b + c \\ \hline 2a + 5b + 2c \\ \hline \end{array}$$

(2.)

$$\begin{array}{r} 4ab - 6a + 3b \\ 3ab - 5a - b \\ \hline ab - a + 4b \\ \hline \end{array}$$

(3.)

$$\begin{array}{r} x^2 + y^2 + z^2 \\ -x^2 - y^2 - z^2 \\ \hline 2x^2 + 2y^2 + 2z^2 \\ \hline \end{array}$$

(4.)

$$\begin{array}{r} 7a - 5b + c \\ a + 2b - 2c \\ \hline \\ \hline \end{array}$$

(5.)

$$\begin{array}{r} 2x^2 + 5xy - 3y^2 \\ 2x^2 - xy - y^2 \\ \hline \\ \hline \end{array}$$

(6.)

$$\begin{array}{r} 5ax - 7by + 3cz \\ 3ax - 6by + 2cz \\ \hline \\ \hline \end{array}$$

(7.)

$$\begin{array}{r} 15\sqrt{x} - 12\sqrt{y} - \sqrt{a} \\ 3\sqrt{x} - 9\sqrt{y} + 3\sqrt{a} \\ \hline \\ \hline \end{array}$$

(8.)

$$\begin{array}{r} a^3 + 3a^2b + 3ab^2 + b^3 \\ a^3 - 3a^2b + 3ab^2 - b^3 \\ \hline \\ \hline \end{array}$$

(9.)

$$\begin{array}{r} -5x^2y + 3xy^2 - 6y^3 \\ 2xy^2 - 6x^2y - 7y^3 \\ \hline \\ \hline \end{array}$$

EXAMPLES FOR PRACTICE.

1. Find the difference of $7x^2 - 8x + 5$, and $5x^2 + 6x - 4$.
   Ans. $2x^2 - 14x + 9$.
2. From $6x - 2a - b + 7$ take $a - 3b + 8 + 4x$.
   Ans. $2x - 3a + 2b - 1$.
3. From $4\ (a + b)^2 - 7\ (c - d)^2$ take $3\ (a + b)^2 - 6\ (c - d)^2$.
   Ans. $(a + b)^2 - (c - d)^2$.
4. From $15x^{\frac{1}{2}} - 10y^{\frac{1}{3}} + 7z^{\frac{1}{4}}$ take $9y^{\frac{1}{3}} - 5z^{\frac{1}{4}} - 3x^{\frac{1}{2}}$.
   Ans. $18x^{\frac{1}{2}} - 19y^{\frac{1}{3}} + 12z^{\frac{1}{4}}$.
5. From $\frac{1}{2}\ (a + b)$ subtract $\frac{1}{2}\ (a - b.)$ Ans. b.
6. From $ax^3 + bx^2 + cx + d$ take $a'x^3 + b'x^2 - c'x + d'$.
   Ans. $(a - a')\ x^3 + (b - b')\ x^2 + (c + c')\ x + (d - d')$.
7. From $3x^n + 7x^{n-1} - 5x^{n-2}$ take $4x^{n-1} - 7x^{n-2} + 3x^{n-3}$.
   Ans. $3x^n + 3x^{n-1} + 2x^{n-2} - 3x^{n-3}$.

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* The term subtraction, used for this rule, is liable to the same objection as that for addition; the operations to be performed being frequently of a mixed nature, like those of the former.

MULTIPLICATION.

Multiplication is the method of finding the product of two or more quantities, and is performed in the same manner as in arithmetic; except that it is usual, in this case, to begin the operation at the left hand, and to proceed towards the right, or contrary to the way employed in the latter.

The rule is commonly divided into three cases; in each of which it is necessary to observe, that like signs, in multiplying, produce +, and unlike signs −.

It is likewise to be remarked, that powers, or roots of the same quantity, are multiplied together by adding their indices: thus, $a \times a^2$, or $a^1 \times a^2 = a^{1+2} = a^3$; $a^2 \times a^3 = a^{2+3} = a^5$; $a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{1}{2}+\frac{1}{3}} = a^{\frac{3}{6}+\frac{2}{6}} = a^{\frac{5}{6}}$; and $a^m \times a^n = a^{m+n}$; where m and n may be either integers or fractions.

The multiplication of compound quantities is also, sometimes, simply denoted by writing them down, with their proper signs, in a bracket, or a parenthesis, without performing the whole operation, as

$$3ab\,(a - b), \text{ or } 2a\sqrt{(a^2 + b^2)}, \text{ or } a\sqrt[3]{(a + b)}.$$

This method is often preferable to that of executing the entire process; particularly when the product of two or more factors is to be divided by some other quantity, because, in this case, any quantity that is common to both the divisor and dividend may be more readily suppressed; as will be evident from various instances in the following part of the work.*

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* The above rule for determining the signs in the product may be thus proved. If the quantity a is greater than the quantity b, and $a - b$ is to be multiplied by c, the product must obviously be less than ac, since $a - b$ is less than a, and consequently when each of the terms of $a - b$ is multiplied by c, the result will be

$$(a - b) \times c = ac - bc,$$

because if it were $ac + bc$, the product would exceed ac, which is absurd.

Also, if a be greater than b, and c greater than d, then we shall have

$$(a - b) \times (c - d) = ac - ad - bc + bd.$$

For the product of $a - b$ by c is $ac - bc$, and that of $a - b$ by $-d$ is $-d\,(a - b)$, as has been before shewn; hence $a - b$ being less than a, it is evident that the part $-d\,(a - b)$ which is to be taken from $c\,(a - b)$ must be less than ad, and since the first term of the product $-d\,(a - b)$ is $-ad$, the second term must be $+bd$; for if it were $-bd$, a part greater than ad would have to be taken from $c\,(a - b)$, which is absurd.

CASE I.

When the factors are both simple quantities.

Rule.—Multiply the coefficients of the two terms together, and to the product annex all the letters, or their powers, belonging to each, after the manner of a word; and the result, with the proper sign prefixed, will be the product required.*

EXAMPLES.

(1.)	(2.)	(3.)	(4.)
$12a$	$-2a$	$+5a$	$-9x^2$
$3b$	$+4b$	$-6x$	$-5bx$
$36ab$	$-8ab$	$-30ax$	$+45bx^3$

(5.)	(6.)	(7.)	(8.)
$7ab$	$-6a^2x$	$-2xy^2$	$-7axy$
$-5ac$	$+5x$	$-xy$	$+6ay$

(9.)	(10.)	(11.)	(12.)
$3a^2b$	$12a^2x$	$-6xyz$	x^{n-1}
$2a^2b$	$-2x^2y$	$+ay^2z$	x^{n+1}

CASE II.

When one of the factors is a compound quantity.

Rule.—Multiply each term of the compound factor, considered as a multiplicand, separately, by the multiplier, as in the former case;

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* When any number of quantities are to be multiplied together, it is the same thing in whatever order they are placed: thus, if ab is to be multiplied by c, the product is either abc, acb, or bca; although it is usual, in this case, as well as in addition and subtraction, to put them according to their rank in the alphabet. It may here also be observed, in conformity to the rule given above for the signs, that $(+a) \times (+b)$ or $(-a) \times (-b) = +ab$,
and $(+a) \times (-b)$ or $(-a) \times (+b) = -ab$.

then these products placed one after another, with their proper signs, will be the whole product required.

EXAMPLES.

(1.)

$$\begin{array}{l} 3a - 2b \\ 4a \\ \hline 12a^2 - 8ab \\ \hline \end{array}$$

(2.)

$$\begin{array}{l} 6xy - 8 \\ 3x \\ \hline 18x^2y - 24x \\ \hline \end{array}$$

(3.)

$$\begin{array}{l} a^2 - 2x + 1 \\ 4x \\ \hline 4a^2x - 8x^2 + 4x \\ \hline \end{array}$$

(4.)

$$\begin{array}{l} 12x - ab \\ 5a \\ \hline \\ \hline \end{array}$$

(5.)

$$\begin{array}{l} 35x^2 - 7a \\ -2x \\ \hline \\ \hline \end{array}$$

(6.)

$$\begin{array}{l} 3y^2 + y - 2 \\ xy \\ \hline \\ \hline \end{array}$$

(7.)

$$\begin{array}{l} 13x^2 - a^2b \\ -2a \\ \hline \\ \hline \end{array}$$

(8.)

$$\begin{array}{l} 25xy + 3a^2 \\ 13x^2 \\ \hline \\ \hline \end{array}$$

(9.)

$$\begin{array}{l} 3x^n - x^{n-1}y^2 - 2y^m \\ 5xy \\ \hline \\ \hline \end{array}$$

CASE III.

When both the factors are compound quantities.

Rule.—Multiply each term of the multiplicand separately, by each term of the multiplier, setting down the products one after another, with their proper signs; then add the several lines of products together, and their sum will be the whole product required.

EXAMPLES.

(1.)

$$\begin{array}{l} a + x \\ a + x \\ \hline a^2 + ax \\ \quad + ax + x^2 \\ \hline a^2 + 2ax + x^2 \\ \hline \end{array}$$

(2.)

$$\begin{array}{l} x - y \\ x - y \\ \hline x^2 - xy \\ \quad - xy + y^2 \\ \hline x^2 - 2xy + y^2 \\ \hline \end{array}$$

(3.)

$$\begin{array}{l} x^2 + xy - y^2 \\ x - y \\ \hline x^3 + x^2y - xy^2 \\ \quad - x^2y - xy^2 + y^3 \\ \hline x^3 \quad \ast \quad - 2xy^2 + y^3 \\ \hline \end{array}$$

$$\begin{array}{lll}
\text{(4.)} & \text{(5.)} & \text{(6.)} \\
\begin{array}{l} a + x \\ a - x \\ \hline a^2 + ax \\ \quad\; - ax - x^2 \\ \hline a^2 \quad * \quad - x^2 \\ \hline \end{array}
&
\begin{array}{l} x^2 - y^2 \\ x^2 + y^2 \\ \hline x^4 - x^2y^2 \\ \quad\; + x^2y^2 - y^4 \\ \hline x^4 \quad * \quad - y^4 \\ \hline \end{array}
&
\begin{array}{l} x^2 + xy + y^2 \\ x \;\, - y \\ \hline x^3 + x^2y + xy^2 \\ \quad\; - x^2y - xy^2 - y^3 \\ \hline x^3 \quad * \qquad * \quad - y^3 \\ \hline \end{array}
\end{array}$$

EXAMPLES FOR PRACTICE.

1. Required the product of $x^2 - xy + y^2$ and $x + y$.
Ans. $x^3 + y^3$.

2. Required the product of $x^3 + x^2y + xy^2 + y^3$ and $x - y$.
Ans. $x^4 - y^4$.

3. Required the product of $x^2 + xy + y^2$ and $x^2 - xy + y^2$.
Ans. $x^4 + x^2y^2 + y^4$.

4. Required the product of $3x^2 - 2xy + 5$ and $x^2 + 2xy - 3$.
Ans. $3x^4 + 4x^3y - 4x^2y^2 - 4x^2 + 16xy - 15$.

5. Required the product of $2a^2 - 3ax + 4r^2$ and $5a^2 - 6ax - 2x^2$.
Ans. $10a^4 - 27a^3x + 34a^2x^2 - 18ax^3 - 8x^4$.

6. Required the product of $5x^3 + 4ax^2 + 3a^2x + a^3$ and $2x^2 - 3ax + a^2$.
Ans. $10x^5 - 7ax^4 - a^2x^3 - 3a^3x^2 + a^5$.

7. Required the product of $3x^3 + 2x^2y^2 + 3y^3$ and $2x^3 - 3x^2y^2 + 5y^3$.
Ans. $6x^6 - 5x^5y^2 - 6x^4y^4 + 21x^3y^3 + x^2y^5 + 15y^6$.

8. Required the product of $x^3 - ax^2 + bx - c$ and $x^2 - dx + e$.
Ans. $x^5 - (a + d)\, x^4 + (b + ad + e)\, x^3 - (c + bd + ae)\, x^2$
$+ (cd + be)\, x - ce$.

9. Multiply $y^7 + y^6 - y^4 - y^3 + y + 1$ by $y^2 - y + 1$.
Ans. $y^9 + 1$

10. Multiply $x^{n-1} + 2x^{n-2} + 3x^{n-3}$ by $x^2 + x + 1$.
Ans. $x^{n+1} + 3x^n + 6x^{n-1} + 5x^{n-2} + 3x^{n-3}$

11. Multiply $x^n + r^{n-1}y^{n-1} + y^n$ by $x^n - x^{n-1}y^{n-1} + y^n$.
Ans. $x^{2n} - x^{2n-2}y^{2n-2} + 2x^n y^n + y^{2n}$.

MULTIPLICATION BY DETACHED COEFFICIENTS.

When the factors consist of powers of the same quantity, the product may be obtained very simply by means of the coefficients alone, and introducing in the result the proper powers of the quantity.

1. Find the product of $x^7 + x^6 - x^4 - x^3 + x + 1$ by $x^2 - x + 1$.

Supplying the absent terms of the series of powers with ciphers, multiplication will be performed in the following manner :—

$$\begin{array}{rrrrrrrrrr}
1 & +1 & +0 & -1 & -1 & +0 & +1 & +1 & & \\
1 & -1 & +1 & & & & & & & \\
\hline
1 & +1 & +0 & -1 & -1 & +0 & +1 & +1 & & \\
 & -1 & -1 & -0 & +1 & +1 & -0 & -1 & -1 & \\
 & & +1 & +1 & +0 & -1 & -1 & +0 & +1 & +1 \\
\hline
1 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +0 & +1
\end{array}$$

Now $x^7 \times x^2 = x^9$; hence the product required is $= x^9 + 1$.

2. Multiply $y^3 + 2y^2 + 2y + 1$ by $y^3 - 2y^2 + 2y - 1$.
Ans. $y^6 - 1$.

3. Multiply together $x - 1$, $x - 3$, $x + 2$, $x + 6$, and $x - 5$.
Ans. $x^5 - x^4 - 37x^3 + 61x^2 + 156x - 180$.

4. Multiply together $y^2 + 4$, $y + 1$, and $y - 1$.
Ans. $y^4 + 3y^2 - 4$.

5. Multiply $x^{-2} + 3x^{-1} + 2 + x$ by $x^2 - 2x + 3$.
Ans. $x^3 + 2x + 1 + 7x^{-1} + 3x^{-2}$.

DIVISION.

Division is the converse of multiplication, and is performed like that of numbers; the rule being usually divided into three cases; in each of which, like signs give + in the quotient, and unlike signs −, as in finding their products.[*]

It is here also to be observed, that powers and roots of the same quantity are divided by subtracting the index of the divisor from that of the dividend. Thus,

$$a^3 \div a^2, \text{ or } \frac{a^3}{a^2} = a^{3-2} = a; \quad a^{\frac{1}{2}} \div a^{\frac{1}{3}}, \text{ or } \frac{a^{\frac{1}{2}}}{a^{\frac{1}{3}}} = a^{\frac{1}{2}-\frac{1}{3}} = a^{\frac{3}{6}-\frac{2}{6}} = a^{\frac{1}{6}};$$

$$a^{\frac{3}{4}} \div a^{\frac{2}{3}} \text{ or } \frac{a^{\frac{3}{4}}}{a^{\frac{2}{3}}} = a^{\frac{3}{4}-\frac{2}{3}} = a^{\frac{9}{12}-\frac{8}{12}} = a^{\frac{1}{12}};$$

$$\text{and } a^m \div a^n, \quad \text{or } \frac{a^m}{a^n} = a^{m-n}.$$

---

[*] According to the rule here given for the signs, it follows that

$$\frac{+ab}{+b} = +a; \frac{-ab}{-b} = +a; \frac{-ab}{+b} = -a; \frac{+ab}{-b} = -a;$$

as will readily appear by multiplying the quotient by the divisor; the signs of the products being then the same as would take place in the former rule.

CASE I.

When the divisor and dividend are both simple quantities.

Rule.—Set the dividend over the divisor, in the manner of a fraction, and reduce it to its simplest form, by cancelling the letters and figures that are common to each term.

EXAMPLES.

$6ab \div 2a$, or $\frac{6ab}{2a} = 3b$; and $12ax^2 \div 3x$, or $\frac{12ax^2}{3x} = 4ax$;

$a \div a$, or $\frac{a}{a} = 1$; and $a \div -a$, or $\frac{a}{-a} = -1$.

Also $-2a \div 3a$, or $\frac{-2a}{3a} = -\frac{2}{3}$;

And $9x^{\frac{1}{2}} \div 3x^{\frac{1}{4}}$, or $\frac{9x^{\frac{1}{2}}}{3x^{\frac{1}{4}}} = 3x^{\frac{1}{2}-\frac{1}{4}} = 3x^{\frac{2}{4}-\frac{1}{4}} = 3x^{\frac{1}{4}}$.

1. Divide $16x^2$ by $8x$, and $12a^2x^2$ by $-8a^2x$.

Ans. $2x$ and $-\frac{3x}{2}$.

2. Divide $-15ay^2$ by $3ay$, and $-18ax^2y$ by $-8ax$.

Ans. $-5y$, and $2\frac{1}{4}xy$.

3. Divide $-\frac{2}{3}a^{\frac{1}{2}}$ by $\frac{4}{5}a^{\frac{1}{2}}$, and $ax^{\frac{1}{3}}$ by $-\frac{3}{5}a^{\frac{1}{2}}x^{\frac{1}{4}}$.

Ans. $-\frac{5}{6}$, and $-\frac{5}{3}a^{\frac{1}{2}}x^{\frac{1}{12}}$.

4. Divide a^6b^5 by a^2b, and x^{m+n} by x^m.

Ans. a^4b^4, and x^n.

5. Divide x^{-5} by x^{-3}, and $a^{-1}b^{-2}c^{-3}$ by abc.

Ans. x^{-2}, and $a^{-2}b^{-3}c^{-4}$.

6. Divide $x^{\frac{1}{m}}$ by $x^{\frac{1}{n}}$. Ans. $x^{\frac{n-m}{mn}}$.

7. Divide x^{2m-1} by x^{m+3}. Ans. x^{m-4}.

8. Divide y^{m+1} by y^{2m-1}. Ans. $y^{-(m-2)} = \frac{1}{y^{m-2}}$.

CASE II.

When the divisor is a simple quantity, and the dividend a compound one.

Rule.—Divide each term of the dividend by the divisor, as in the former case, setting down such as will not divide in the simplest form they will admit of.

EXAMPLES.

$(ab+b^2) \div 2b$, or $\dfrac{ab+b^2}{2b} = \frac{1}{2}a + \frac{1}{2}b = \dfrac{a+b}{2}$.

$(10ab - 15ax) \div 5a$, or $\dfrac{10ab - 15ax}{5a} = 2b - 3x$.

$(30ax - 48x^2) \div 6x$, or $\dfrac{30ax - 48x^2}{6x} = 5a - 8x$.

1. Let $3x^3 + 6x^2 + 3ax - 15x$ be divided by $3x$.
Ans. $x^2 + 2x + a - 5$.

2. Let $3abc + 12abx - 9a^2b$ be divided by $3ab$.
Ans. $c + 4x - 3a$.

3. Let $40a^3b^3 + 60a^2b^2 - 17ab$ be divided by $-ab$.
Ans. $-40a^2b^2 - 60ab + 17$.

4. Let $15a^2bc - 12acx^2 + 5ad^2$ by divided by $-5ac$.
Ans. $-3ab + \dfrac{12}{5}x^2 - \dfrac{d^2}{c}$.

5. Let $20ax + 15\,ax^2 + 10ax + 5a$ be divided by $5a$.
Ans. $3x^2 + 6x + 1$.

6. Divide $x^n + x^{n+1} + x^{n+2} + x^{n+3}$ by x^n.
Ans. $1 + x + x^2 + x^3$.

7. Divide $x^{m-3} + ax^{m-2} + bx^{m-1}$ by x^{m-4}.
Ans. $x + ax^2 + bx^3$.

8. Divide $a^nx - a^{n+1}x^2 + a^{n+2}x^3 - a^{n+3}x^4$ by a^n.
Ans. $x - ax^2 + a^2x^3 - a^3x^4$.

CASE III.

When the divisor and dividend are both compound quantities.

1. Set the quantities down in the same manner as in division of numbers, ranging the terms of each of them so that the higher powers of one of the letters may stand before the lower.

2. Divide the first term of the dividend by the first term of the divisor, and set the result in the quotient, with its proper sign, or simply by itself, if it be affirmative or positive.

3. Multiply the whole divisor by the term thus found; and having subtracted the result from the dividend, bring down as many terms to the remainder as are requisite for the next operation, which perform as before; and so on, till the last term of the dividend is brought down.

EXAMPLES.

$$\begin{array}{l}
x + y)x^2 + 2xy + y^2(x + y \\
\quad\quad x^2 + xy \\
\hline
\quad\quad\quad xy + y^2 \\
\quad\quad\quad xy + y^2 \\
\hline
\quad\quad\quad\quad * \\
\hline
\end{array}$$

$$\begin{array}{l}
a + x)a^3 + 5a^2x + 5ax^2 + x^3(a^2 + 4ax + x^2 \\
\quad\quad a^3 + a^2x \\
\hline
\quad\quad\quad 4a^2x + 5ax^2 \\
\quad\quad\quad 4a^2x + 4ax^2 \\
\hline
\quad\quad\quad\quad\quad ax^2 + x^3 \\
\quad\quad\quad\quad\quad ax^2 + x^3 \\
\hline
\quad\quad\quad\quad\quad\quad * \\
\hline
\end{array}$$

$$\begin{array}{l}
x - 3)x^3 - 9x^2 + 27x - 27(x^2 - 6x + 9 \\
\quad\quad x^3 - 3x^2 \\
\hline
\quad\quad\quad - 6x^2 + 27x \\
\quad\quad\quad - 6x^2 + 18x \\
\hline
\quad\quad\quad\quad\quad 9x - 27 \\
\quad\quad\quad\quad\quad 9x - 27 \\
\hline
\quad\quad\quad\quad\quad\quad * \\
\hline
\end{array}$$

$$\begin{array}{l}
2x^2 - 3ax + a^2)4x^4 - 9a^2x^2 + 6a^3x - a^4(2x^2 + 3ax - a^2 \\
\quad\quad\quad\quad 4x^4 - 6ax^3 + 2a^2x^2 \\
\hline
\quad\quad\quad\quad\quad\quad 6ax^3 - 11a^2x^2 + 6a^3x \\
\quad\quad\quad\quad\quad\quad 6ax^3 - 9a^2x^2 + 3a^3x \\
\hline
\quad\quad\quad\quad\quad\quad\quad\quad - 2a^2x^2 + 3a^3x - a^4 \\
\quad\quad\quad\quad\quad\quad\quad\quad - 2a^2x^2 + 3a^3x - a^4 \\
\hline
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad * \\
\hline
\end{array}$$

Note 1. If the divisor be not exactly contained in the dividend, the quantity that remains after the division is finished must be placed over the divisor at the end of the quotient in the form of a fraction: Thus *

$$\begin{array}{l}
a + x)\,a^3 - x^3\,(a^2 - ax + x^2 - \dfrac{2x^3}{a+x} \\
\qquad a^3 + a^2x \\
\hline
\qquad\quad - a^2x - x^3 \\
\qquad\quad - a^2x - ax^2 \\
\hline
\qquad\qquad\qquad ax^2 - x^3 \\
\qquad\qquad\qquad ax^2 + x^3 \\
\hline
\qquad\qquad\qquad\qquad - 2x^3 \\
\hline
\end{array}$$

$$\begin{array}{l}
x + y)\,x^4 + y^4\,(x^3 - x^2y + xy^2 - y^3 + \dfrac{2y^4}{x+y} \\
\qquad x^4 + x^3y \\
\hline
\qquad\quad - x^3y + y^4 \\
\qquad\quad - x^3y - x^2y^2 \\
\hline
\qquad\qquad\qquad x^2y^2 + y^4 \\
\qquad\qquad\qquad x^2y^2 + xy^3 \\
\hline
\qquad\qquad\qquad\qquad - xy^3 + y^4 \\
\qquad\qquad\qquad\qquad - xy^3 - y^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad 2y^4 \\
\hline
\end{array}$$

2. The division of quantities may also sometimes be carried on, *ad infinitum*, like a decimal fraction; in which case a few of the leading terms of the quotient will generally be sufficient to indicate the rest, without its being necessary to continue the operation: Thus,

---

* In the case here given, the operation of division may be considered as terminated, when the highest power of the letter, in the first, or leading term of the remainder, by which the process is regulated, is less than the power of the first term of the divisor; or when the first term of the divisor is not contained in the first term of the remainder; as the succeeding part of the quotient, after this, instead of being integral as it ought to be, would necessarily become fractional.

$$\begin{array}{l}
a + x)\,a \ldots .\,(1 - \dfrac{x}{a} + \dfrac{x^2}{a^2} - \dfrac{x^3}{a^3} + \dfrac{x^4}{a^4} - \&c. \\
\quad\;\; \underline{a + x} \\
\quad\;\; - x \\
\quad\;\; \underline{- x - \dfrac{x^2}{a}} \\
\qquad\quad + \dfrac{x^2}{a} \\
\qquad\quad \underline{+ \dfrac{x^2}{a} + \dfrac{x^3}{a^2}} \\
\qquad\qquad\quad - \dfrac{x^3}{a^2} \\
\qquad\qquad\quad \underline{- \dfrac{x^3}{a^2} - \dfrac{x^4}{a^3}} \\
\qquad\qquad\qquad\quad \underline{+ \dfrac{x^4}{a^3}}
\end{array}$$

And by a process similar to the above it may be shewn that

$$\frac{a}{a - x} = 1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3} + \frac{x^4}{a^4} + \frac{x^5}{a^5} + \&c.$$

Whence the law, by which either of these series may be continued at pleasure, is obvious.

EXAMPLES FOR PRACTICE.

1. Let $a^2 - 2ax + x^2$ be divided by $a - x$.
   Ans. $a - x$.
2. Let $x^3 - 3ax^2 + 3a^2x - a^3$ be divided by $x - a$.
   Ans. $x^2 - 2ax + a^2$.
3. Let $a^3 + 5a^2x + 5ax^2 + x^3$ be divided by $a + x$.
   Ans. $a^2 + 4ax + x^2$.
4. Let $2y^3 - 19y^2 + 26y - 17$ be divided by $y - 8$.
   Ans. $2y^2 - 3y + 2 - \dfrac{1}{y - 8}$.
5. Divide $x^5 + 1$ by $x + 1$, and $x^6 - 1$ by $x - 1$.
   Ans. $x^4 - x^3 + x^2 - x + 1$, and $x^5 + x^4 + x^3 + x^2 + x + 1$.
6. Divide $48x^3 - 76ax^2 - 64a^2x + 105a^3$ by $2x - 3a$.
   Ans. $24x^2 - 2ax - 35a^2$.

7. Let $4x^4 - 9x^2 + 6x - 3$ be divided by $2x^2 + 3x - 1$.

Ans. $2x^2 - 3x + 1 - \frac{2}{2x^2 + 3x - 1}$.

8. Let $x^4 - a^2x^2 + 2a^3x - a^4$ be divided by $x^2 - ax + a^2$.

Ans. $x^2 + ax - a^2$.

9. Let $6x^4 - 96$ be divided by $3x - 6$, and $a^5 + x^5$ by $a + x$.

Ans. $2x^3 + 4x^2 + 8x + 16$, and $a^4 - a^3x + a^2x^2 - ax^3 + x^4$.

10. Let $32x^5 + 243$ be divided by $2x + 3$, and $x^6 - a^6$ by $x - a$.

Ans. $16x^4 - 24x^3 + 36x^2 - 54x + 81$, and $x^5 + ax^4 + a^2x^3 + a^3x^2 + a^4x + a^5$.

11. Let $b^4 - 3y^4$ be divided by $b - y$, and $a^4 + 16ab^3 + 16b^4$ by $a + 2b$.

Ans. $b^3 + b^2y + by^2 + y^3 - \frac{2y^4}{b + y}$, and $a^3 - 2a^2b + 4ab^2 + 8b^3$.

12. Let $x^2 + px + q$ be divided by $x + a$, and $x^3 - px^2 + qx - r$ by $x - a$.

Ans. $x + (p - a) + \frac{a^2 - pa + q}{x + a}$, and

$$x^2 + (a - p)\, x + (a^2 - ap + q) + \frac{a^3 - pa^2 + qa - r}{x - a}.$$

13. Divide $a^3x^3 - 2ab^2x + b^3$ by $ax - b$.

Ans. $a^2x^2 + abx - b^2$.

14. Divide $mnx^3 - (n^2 + mr)\, x^2 + 2nrx - r^2$ by $nx - r$.

Ans. $mx^2 - nx + r$.

15. Divide $a - x$ by $a^{\frac{1}{n}} - x^{\frac{1}{n}}$.

Ans. $a^{\frac{n-1}{n}} + a^{\frac{n-2}{n}} . x^{\frac{1}{n}} + a^{\frac{n-3}{n}} . x^{\frac{2}{n}} \ldots . + a^{\frac{1}{n}} . x^{\frac{n-2}{n}} + x^{\frac{n-1}{n}}$

16. Divide $a^3 + x^3$ by $a + x$.

Ans. $a^2 - ax + x^2$.

17. Divide $x^4 - y^4$ by $x + y$.

Ans. $x^3 - x^2y + xy^2 - y^3$.

18. By a similar process, shew that $a^n - x^n$ is divisible by $a + x$ when n is *even*, $a^n + x^n$ by $a + x$ when n is *odd*; and if n be *even*, $a^n + x^n$ is divisible by *neither* $a + x$ *nor* $a - x$.

19. Shew that $a^n - x^n$ is divisible by $a - x$ whether n is *even* or *odd*.

DIVISION BY DETACHED COEFFICIENTS.

The method of performing division by detached coefficients may frequently be employed with much advantage, and, in illustration of the method, we have subjoined an example or two.

1. Divide $x^6 - 2x^3 + 1$ by $x^2 - 2x + 1$.

$$
\begin{array}{l}
1-2+1)1+0+0-2+0+0+1(1+2+3+2+1\\
1-2+1\\
\hline
2-1-2\\
2-4+2\\
\hline
+3-4+0\\
3-6+3\\
\hline
+2-3+0\\
2-4+2\\
\hline
1-2+1\\
1-2+1
\end{array}
$$

Now $x^6 \div x^2 = x^4$; hence

quotient $= x^4 + 2x^3 + 3x^2 + 2x + 1$.

2. Divide $32a^5 + 243$ by $2a + 3$.
Ans. $16a^4 - 24a^3 + 36a^2 - 54a + 81$.

3. Divide $a^5 - 3a^4 + a^3 + a^2 - 3a + 1$ by $a + 1$.
Ans. $a^4 - 4a^3 + 5a^2 - 4a + 1$.

SYNTHETIC DIVISION.

The synthetic method of division is still more useful than the method of detached coefficients. It is simply a change of arrangement of the work, and the conversion of the process of subtraction into that of addition. This conversion is effected by simply changing the signs of all the terms of the divisor except that of the first term. The process will be understood from an attentive consideration of the following examples:—

1. Divide $6x^5 + 2x^4 - 21x^3 + 12x^2 - 21x + 8$ by $x^2 + 2x - 1$.

By detached coefficients.

$$\begin{array}{lrrrrrrl}
\overset{x^2}{1+2-1}\big) & \overset{x^5}{6} & +\ 2 & -\ 21 & +\ 12 & -\ 21 & +\ 8 & \big(\overset{x^3}{6} - 10 + 5 - 8 \\
 & 6 & +\ 12 & -\ 6 & & & & \therefore 6x^3 - 10x^2 + 5x - 8 \\
\cline{2-4}
 & & -\ 10 & -\ 15 & +\ 12 & & & = \text{quotient.} \\
 & & -\ 10 & -\ 20 & +\ 10 & & & \\
\cline{3-5}
 & & & 5 & +\ 2 & -\ 21 & & \\
 & & & 5 & +\ 10 & -\ 5 & & \\
\cline{4-6}
 & & & & -\ 8 & -\ 16 & +\ 8 & \\
 & & & & -\ 8 & -\ 16 & +\ 8 & \\
\cline{5-7}
\end{array}$$

By the synthetic method.

$$\begin{array}{r|rrrrrr}
 & 6 & +\ 2 & -\ 21 & +\ 12 & -\ 21 & +\ 8 \\
-\ 2 & & -\ 12 & +\ 20 & -\ 10 & +\ 16 & \\
+\ 1 & & & +\ 6 & -\ 10 & +\ 5 & -\ 8 \\
\hline
 & 6 & -\ 10 & +\ 5 & -\ 8 & * & *
\end{array}$$

$\therefore 6x^3 - 10x^2 + 5x - 8 =$ quotient.

The student will easily perceive that the latter process is only a modified arrangement of the work in the preceding method by detached coefficients. In the former, the coefficients are 2 and — 1, and in the latter they are changed into — 2 and + 1, and multiplying by the coefficient of the first term in the quotient, which is 6, the products — 12 and + 6 are written diagonally instead of horizontally. Then the *sum* of — 12 and 2 is — 10, the coefficient of the second term in the quotient. Again, multiplying by — 10, and writing the products + 20 and — 10 diagonally as before, the sum of — 21, + 20 and + 6 is 5, the coefficient of the third term of the quotient, and so on till the process either terminates, or leaves a remainder, which may be written over the divisor in the form of a fraction, and thus the quotient will be completely determined.

When the coefficient of the first term of the divisor is unity, the process is very simple, as is evident from the preceding example;

but when that coefficient is not unity, the process might be managed as in the following example :—

2. Divide $72x^4 - 78x^3 - 10x^2 + 18x + 6$ by $6x^2 - 4x - 1$.

$$\begin{array}{r|ccc|ccl}
x^2 & x^4 & x^3 & x^2 & x & & \\
6 & 72 & -78 & -10 & +18 & +6 & \\
+4 & & +48 & -20 & -12 & & \\
+1 & & & +12 & -5 & -3 & \\
\hline
\text{Divide by } 6 & 72 & -30 & -18 & +1 & +3 & = \text{remainder.} \\
\cline{2-4}
\text{quotient} = & 12 & -5 & -3 & & &
\end{array}$$

Hence the complete quotient $= 12x^2 - 5x - 3 + \dfrac{x + 3}{6x^2 - 4x - 1}$.

EXAMPLES FOR PRACTICE.

1. Divide $a^6 - 5a^5 + 15a^4 - 24a^3 + 27a^2 - 13a + 5$ by $a^2 - 3a + 5$.
   Ans. $a^4 - 2a^3 + 4a^2 - 2a + 1$.
2. Divide $1 - y$ by $1 + y$.
   Ans. $1 - 2y + 2y^2 - 2y^3 + \ldots\ldots$
3. Divide $1 + 2x$ by $1 - x - x^2$.
   Ans. $1 + 3x + 4x^2 + 7x^3 + 11x^4 + \ldots\ldots$
4. Divide $3x^4 - 4x^3 - 6x^2 + 16x - 24$ by $x - 2$, and also by $x + 2$.
   Ans. $\begin{cases} 3x^3 + 2x^2 - 2x + 12. \\ 3x^3 - 10x^2 + 14x - 12. \end{cases}$
5. Divide $6y^5 - 32y^4 + 77y^3 - 86y^2 - 66y + 216$ by $4y^2 - 8y + 24$.
   Ans. $\frac{3}{2}y^3 - 5y^2 + \frac{1}{4}y + 9$.

ALGEBRAIC FRACTIONS.

Algebraic Fractions have the same names and rules of operation as numerical fractions in common arithmetic; and the methods of reducing them, in either of these branches, to their most convenient forms, depend upon similar principles.

CASE I.

THE GREATEST COMMON MEASURE.

A *measure* of a quantity is one which divides that quantity without remainder.

A *common measure* of two or more quantities is one which divides each of these quantities without remainder.

The *greatest common measure* of two or more quantities is the greatest of all the quantities which divides each of the proposed quantities without remainder.

To find the greatest common measure of two algebraic quantities.

Rule. 1. Arrange the two quantities according to the powers of some common letter, and divide that which contains the highest exponent of the common letter by the other, or if the highest exponent be the same in each of the quantities, take either of them for a dividend.

2. Divide this divisor by the remainder, and repeat this operation till there is *no* remainder: the *last divisor* is the greatest common measure.

3. If each of the terms of one of the quantities contain a factor which is not contained in each of the terms of the other, it may be rejected previous to the division, because it is no part of the common measure, and a similar modification may be made whenever, in the course of the operation, it can be effected.

4. If the sign of the first term of any divisor be negative, the signs of all the terms of the divisor may be changed without affecting the result; and if the numerical coefficient of the first term of the original, or of any new dividend, be not divisible by that of the first term of the corresponding divisor, the several terms of any such dividend may be multiplied by such a number or quantity as will give an integral quotient. This artifice avoids fractional quotients. The greatest common measure of monomial quantities is easily found by inspection: thus, the greatest common measure of $12a^2b^2$ and $20a^3b$ is $4a^2b$.

5. The greatest common measure of three or more quantities is found by first finding the greatest common measure of two of them, and then of that common measure and a third, and so on.

EXAMPLES.

1. Find the greatest common measure of $6a^2b^2$ and $2a^3b-6ab^3$.

$$\text{Here } 2a^3b - 6ab^3 = 2ab(a^2 - 3b^2), \text{ and } 6a^2b^2 = 2ab \times 3ab;$$

$$\therefore \text{greatest common measure} = 2ab.$$

2. Find the greatest common measure of $6a^2 + 7a - 3$ and $12a^3 - 19a^2 + 14a - 3$.

$$\begin{array}{lrl}
 & 6a^2 + 7a - 3) & 12a^3 - 19a^2 + 14a - 3(2a - 11 \\
 & & 12a^3 + 14a^2 - 6a \\ \cline{3-3}
 & & \quad - 33a^2 + 20a - 3 \\
\text{Multiply by} & & \quad 2 \\ \cline{3-3}
 & & \quad - 66a^2 + 40a - 6 \\
 & & \quad - 66a^2 - 77a + 33 \\ \cline{3-3}
\text{Divide by} & & 39)117a - 39 \\ \cline{3-3}
 & & \qquad 3a - 1)6a^2 + 7a - 3(2a + 3 \\
 & & \qquad\qquad 6a^2 - 2a \\ \cline{3-3}
 & & \qquad\qquad\quad + 9a - 3 \\
 & & \qquad\qquad\quad 9a - 3 \\ \cline{3-3}
\end{array}$$

Hence greatest common measure $= 3a - 1$.

3. Find the greatest common measure of the two quantities,

$2a^4 + a^3 - 20a^2 - 7a + 24$ and $2a^4 + 3a^3 - 13a^2 - 7a + 15$.

The labour of writing may be abridged by using detached coefficients, and the process may be neatly arranged by taking the successive divisors on the left and right alternately of the dividends, as in the following work :—

1	2 + 1 − 20 − 7 + 24 2 + 7 + 0 − 9	2 + 3 − 13 − 7 + 15 2 + 1 − 20 − 7 + 24	1
−3	− 6 − 20 + 2 + 24 − 6 − 21 + 0 + 27	2 + 7 + 0 − 9 2 + 4 − 6	2
	1 + 2 − 3 coefficients of last divisor.	3 + 6 − 9 3 + 6 − 9	3

Hence $a^2 + 2a - 3 =$ the greatest common measure.

EXAMPLES FOR PRACTICE.

Find the greatest common measure of the quantities in the following examples :—

1. $a^2 + 2a + 1$ and $a^3 - 2a - 1$. Ans. $a + 1$.
2. $x^4 - 1$ and $x^5 + x^3$. Ans. $x^2 + 1$.
3. $x^3 - b^2x$ and $x^2 + 2bx + b^2$. Ans. $x + b$.
4. $3a^2 - 2a - 1$ and $4a^3 - 2a^2 - 3a + 1$. Ans. $a - 1$.
5. $x^3 - a^3$ and $x^4 - a^4$. Ans. $x - a$.
6. $x^5 + x^3y^2 + x^2y^3 + y^5$ and $x^4 - y^4$. Ans. $x^3 + x^2y + xy^2 + y^3$.
7. $a^4 - x^4$ and $a^3 - a^2x - ax^2 + x^3$. Ans. $a^2 - x^2$.
8. $x^4 + a^2x^2 + a^4$ and $x^4 + ax^3 - a^3x - a^4$. Ans. $x^2 + ax + a^2$.
9. $7a^2 - 23ab + 6b^2$ and $5a^3 - 18a^2b + 11ab^2 - 6b^3$. Ans. $a - 3b$.
10. $x^2 - 8x + 7$, $x^2 + 4x - 5$, and $x^3 - 9x^2 + 23x - 15$. Ans. $x - 1$.
11. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ and $a^2 - b^2 - c^2 - 2bc$. Ans. $a + b + c$.

CASE II.

THE LEAST COMMON MULTIPLE.

A *multiple* of a quantity is one which can be divided by that quantity without remainder.

A *common multiple* of two or more quantities is a quantity which can be divided by each of these quantities without remainder.

The *least common multiple* of two or more quantities is the *least* quantity which is divisible by each of them without remainder.

To find the least common multiple of two algebraical quantities.

Rule.—Divide the product of the two quantities by their greatest common measure, and the quotient is their least common multiple.

Or, divide *either* of the two quantities by their greatest common measure, and multiply the result by the other quantity.

The least common multiple of three quantities is found by first finding the least common multiple of any two of them, and then of that common multiple and the third, and so on for any number of quantities.

EXAMPLES.

1. Find the least common multiple of $9x^2$ and $6ax$.

The greatest common measure is $3x$; hence

$$\frac{9x^2 \times 6ax}{3x} = 3x \times 6ax = 18ax^2$$

$$\text{or } \frac{9x^2 \times 6ax}{3x} = 9x^2 \times 2a = 18ar^2 = \text{least common multiple.}$$

2. Find the least common multiple of $6a^2 - a - 1$ and $2a^2 + 3a - 2$.

1	2 + 3 − 2 2 − 1	6 − 1 − 1 6 + 9 − 6	3
2	4 − 2 4 − 2	−10 + 5 or 2 − 1	

Hence $2a - 1$ is the greatest common measure, and if one of the quantities, as $2a^2 + 3a - 2$, be divided by $2a - 1$, the quotient is $a + 2$, and therefore $(a + 2)\ (6a^2 - a - 1) = 6a^3 + 11a^2 - 3a - 2$, the least common multiple.

EXAMPLES FOR PRACTICE.

Find the least common multiple of the quantities in the following examples :—

1. $2a^2$, $6ab^3$, and $4a^3b^2$. Ans. $12a^3b^3$.
2. a^3, $3ab^2$, $6bc$ and axy. Ans. $6a^3b^2cxy$.
3. $2a^4$, $5a^3b$, and $6a^2b^2$. Ans. $30a^4b^2$.
4. $x^2 - 8x + 7$ and $x^2 + 7x - 8$. Ans. $x^3 - 57x + 56$.
5. $a^3 - a^2b - ab^2 + b^3$ and $a^3 - a^2b + ab^2 - b^3$. Ans. $a^5 - a^4b - ab^4 + b^5$.
6. $a + x$, $a^2 - x^2$, and $a^2 - 2ax + x^2$. Ans. $a^3 - a^2x - ax^2 + x^3$.
7. $x^2 - 1$, $x^3 + x^2 + x + 1$, and $x^3 - x^2 + x - 1$. Ans. $x^4 - 1$.
8. $y^3 - 4y^2 + 9y - 10$, and $y^3 + 2y^2 - 3y + 20$. Ans. $y^4 - 7y^2 + 26y - 40$.

CASE III.

REDUCTION OF ALGEBRAIC FRACTIONS.

To reduce a fraction to its lowest or most simple terms.

Rule.—Divide the terms of the fraction by the greatest number or quantity that will divide each of them without remainder; or find their greatest common measure, and divide the numerator and denominator by that common measure.

EXAMPLES.

1. Reduce $\dfrac{a^2bc}{5a^2b^2}$ and $\dfrac{x^2}{ax + x^2}$ to their lowest terms.

$$\frac{a^2bc}{5a^2b^2} = \frac{a^2b \times c}{a^2b \times 5b} = \frac{c}{5b}. \qquad \frac{x^2}{ax + x^2} = \frac{x \times x}{x(a + x)} = \frac{x}{a + x}.$$

2. Reduce $\frac{cx+x^2}{a^2c+a^2x}$ to its lowest terms.

$$\frac{cx+x^2}{a^2c+a^2x} = \frac{x\,(c+x)}{a^2\,(c+x)} = \frac{x}{a^2} = \text{fraction required.}$$

3. Reduce $\frac{a^2+a-12}{a^3-5a^2+7a-3}$ to its lowest terms.

$$\begin{array}{r|l|l|l}
1 & 1+1-12 & 1-5+7-3 & 1 \\
 & 1-3 & 1+1-12 & \\
\hline
+4 & +4-12 & -6+19-3 & -6 \\
 & 4-12 & -6-6+72 & \\
\hline
 & & 25\,)\,25-75 & \\
 & & 1-3 &
\end{array}$$

Hence $a-3$ is the greatest common measure, and

$$\frac{a^2+a-12}{a^3-5a^2+7a-3} = \frac{(a^2+a-12)\div(a-3)}{(a^3-5a^2+7a-3)\div(a-3)} = \frac{a+4}{a^2-2a+1}.$$

EXAMPLES FOR PRACTICE.

	Answers.
1. $\frac{x^3-b^2x}{x^2+2bx+b^2}$.	$\frac{x^2-bx}{x+b}$.
2. $\frac{x^4-a^4}{x^5-a^2x^3}$.	$\frac{x^2+a^2}{x^3}$.
3. $\frac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$.	$\frac{3a-x}{3a+x}$.
4. $\frac{2x^3-16x-6}{3x^3-24x-9}$.	$\frac{2}{3}$.
5. $\frac{x^2-9x+20}{x^2+6x-55}$.	$\frac{x-4}{x+11}$.
6. $\frac{9x^5+2x^3+4x^2-x+1}{15x^4-2x^3+10x^2-x+2}$.	$\frac{3x^3+x^2+1}{5x^2+x+2}$.
7. $\frac{3a^3-3a^2b+ab^2-b^3}{4a^2-ab-3b^2}$.	$\frac{3a^2+b^2}{4a+3b}$.
8. $\frac{x^2+y^2-z^2-2xy}{x^2-y^2-z^2-2yz}$.	$\frac{x-y+z}{x+y+z}$.
9. $\frac{x^4-1}{x^6-1}$	$\frac{x^2+1}{x^4+x^2+1}$.
10. $\frac{a^{3m}+a^{2m}-2}{a^{2m}+a^m-2}$.	$\frac{a^{2m}+2a^m+2}{a^m+2}$.

CASE IV.

To reduce a mixed quantity to the form of a fraction.

Any integral quantity, a, may be reduced to the form of a fraction by writing 1 for the denominator. Thus $a = \frac{a}{1}$; and it may be put in a fractional form, having any specified denominator, b, by multiplying both a and 1 by b; thus $a = \frac{a}{1} = \frac{ab}{b}$.

Now, if $a + \frac{b}{c}$ be the mixed quantity, composed of an integral quantity a, and a fraction $\frac{b}{c}$; then

$$a + \frac{b}{c} = \frac{a}{1} + \frac{b}{c} = \frac{ac}{c} + \frac{b}{c} = \frac{ac + b}{c}.$$

Also, if $a - \frac{b}{c}$ be the mixed quantity, then $a - \frac{b}{c} = \frac{ac - b}{c}$; hence we have the following

Rule.—Multiply the integral part by the denominator of the fraction, and to the product add the numerator when it is positive, or subtract it when it is negative, then the result placed over the denominator, will be the fraction required.

EXAMPLES.

1. Reduce $3\frac{2}{5}$ and $x + \frac{a}{x}$ to fractions.

$$3\tfrac{2}{5} = \frac{3 \times 5 + 2}{5} = \frac{15 + 2}{5} = \frac{17}{5}.$$

$$x + \frac{a}{x} = \frac{x \times x + a}{x} = \frac{x^2 + a}{x}.$$

2. Reduce $a + b - \frac{a^2 - b^2 - c^2}{a - b}$ to the form of a fraction.

$$a + b - \frac{a^2 - b^2 - c^2}{a - b} = \frac{(a + b)(a - b) - (a^2 - b^2 - c^2)}{a - b}$$

$$= \frac{a^2 - b^2 - a^2 + b^2 + c^2}{a - b} = \frac{c^2}{a - b}.$$

3. Reduce $1 - \frac{2x}{a}$ and $5a - \frac{3x - b}{a}$ to fractions.

Ans. $\frac{a - 2x}{a}$ and $\frac{5a^2 - 3x + b}{a}$.

4. Reduce $x - \dfrac{ax + x^2}{2a}$ and $5 + \dfrac{2x - 7}{3x}$ to fractions.

Ans. $\dfrac{ax - x^2}{2a}$ and $\dfrac{17x - 7}{3x}$.

5. Reduce $1 - \dfrac{x - a - 1}{a}$ and $1 + 2x - \dfrac{x - 3}{5x}$ to fractions.

Ans. $\dfrac{2a + 1 - x}{a}$ and $\dfrac{10x^2 + 4x + 3}{5x}$.

6. Reduce $\dfrac{a^2 + b^2}{2ab} + 1$ and $\dfrac{a^2 + b^2}{2ab} - 1$ to fractions.

Ans. $\dfrac{(a + b)^2}{2ab}$ and $\dfrac{(a - b)^2}{2ab}$.

7. Reduce $1 + \dfrac{a^2 + b^2 - c^2}{2ab}$ and $1 - \dfrac{a^2 + b^2 - c^2}{2ab}$ to fractions.

Ans. $\dfrac{(a + b + c)(a + b - c)}{2ab}$ and $\dfrac{(a - b + c)(-a + b + c)}{2ab}$.

CASE V.

To reduce a fraction to a whole or mixed quantity.

The fraction $\dfrac{ac + b}{c} = \dfrac{ac}{c} + \dfrac{b}{c} = a + \dfrac{b}{c}$, and the fraction $\dfrac{ac - b}{c}$ $= a - \dfrac{b}{c}$; hence the following

Rule.—Divide the numerator by the denominator for the integral part, and place the remainder, if any, over the denominator for the fractional part; then the fractional part added to the integral part, when the remainder is positive, or subtracted from it when negative, will give the mixed quantity required.

EXAMPLES.

1. Reduce $\dfrac{27}{5}$ and $\dfrac{ax - x^2}{a}$ to whole or mixed quantities.

$$\frac{27}{5} = 5\tfrac{2}{5}, \text{ and } \frac{ax - x^2}{x} = \frac{ax}{x} - \frac{x^2}{x} = a - x.$$

2. Reduce $\dfrac{ab - 2a^2}{ab}$ and $\dfrac{a^2 + x^2}{a - x}$ to mixed quantities.

Ans. $1 - \dfrac{2a}{b}$ and $a + x + \dfrac{2x^2}{a - x}$.

3. Reduce $\frac{x^3 - y^3}{x - y}$ and $\frac{x^3 + y^3}{x + y}$ to whole quantities.

Ans. $x^2 + xy + y^2$, and $x^2 - xy + y^2$.

4. Reduce $\frac{10x^2 - 5x + 3}{5x}$ and $\frac{4x^2 - 8x + 3}{4x}$ to mixed quantities.

Ans. $2x - 1 + \frac{3}{5x}$, and $x - 2 + \frac{3}{4x}$.

5. Reduce $\frac{3x^2 + 6x + 5}{x + 4}$ and $\frac{2x^4 + 19x^2 + 35}{x^3 - 3x^2 + 7x - 21}$ to mixed quantities.

Ans. $3x - 6 + \frac{29}{x + 4}$, and $2x + 6 + \frac{23}{x - 3}$.

6. Reduce $\frac{(a - 2x)^3}{a - x}$ and $\frac{x^{pq} - 1}{x^p - 1}$ to whole or mixed quantities.

Ans. $a^2 - 5ax + 7x^2 - \frac{x^3}{a - x}$ and $x^{p(q-1)} + x^{p(q-2)} + ... + x^{2p} + x^p + 1$.

CASE VI.

To reduce fractions having different denominators to equivalent fractions that shall have a common denominator.

Since $\frac{a}{b} = \frac{a \times d}{b \times d} = \frac{ad}{bd}$, and $\frac{c}{d} = \frac{c \times b}{d \times b} = \frac{bc}{bd}$; therefore $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent to $\frac{ad}{bd}$ and $\frac{bc}{bd}$ which have a common denominator bd; hence we have this rule:

Rule. 1. Multiply each numerator by all the denominators except its own for the new numerators, and all the denominators together for a common denominator.

2. When two or more of the denominators have a common measure, find the least common multiple of the denominators, and it will be the least common denominator.

3. Divide this common denominator by the denominator of one of the given fractions, then the numerator of the fraction being multiplied by the quotient thus obtained, will give the new numerator of that fraction, and so on for all the fractions.

EXAMPLES.

1. Reduce the fractions $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$, and $\frac{h}{k}$, to equivalent fractions having a common denominator.

$$\left.\begin{array}{lcl} a\times d\times f\times k & = & adfk \\ c\times b\times f\times k & = & bcfk \\ e\times b\times d\times k & = & bdek \\ h\times b\times d\times f & = & bdfh \end{array}\right\}\text{ the respective new numerators,}$$

$$b\times d\times f\times k \;=\; bdfk = \text{the common denominator,}$$

hence the equivalent fractions are

$$\frac{adfk}{bdfk},\ \frac{bcfk}{bdfk},\ \frac{bdek}{bdfk},\ \text{and}\ \frac{bdfh}{bdfk}.$$

2. Reduce $\frac{a}{12}, \frac{b}{15}, \frac{2c}{5}, \frac{3d}{7x}, \frac{4e}{21x}$, and $\frac{5h}{63x}$ to equivalent fractions having a common denominator.

Here we may reject 5, $7x$ and $21x$, because 15 is a multiple of 5 and $63x$ is a common multiple of $7x$ and $21x$, and writing the remaining denominators in a line, we have the following process:

$$\begin{array}{r|l} 3 & 12,\quad 15,\quad 63x \\ \hline & 4,\quad 5,\quad 21x, \end{array}$$

$$\therefore\ 3\times 4\times 5\times 21x = 1260x = \text{the least common denominator.}$$

$$\begin{array}{lcccl} & & & \text{New Numerators.} & \\ \textbf{Hence}\ a\times\frac{1260x}{12} & = a\times 105x & = & 105ax & \therefore\ \frac{a}{12} = \frac{105ax}{1260x} \\ b\times\frac{1260x}{15} & = b\times 84x & = & 84bx & \frac{b}{15} = \frac{84bx}{1260x} \\ 2c\times\frac{1260x}{5} & = 2c\times 252x & = & 504cx & \frac{2c}{5} = \frac{504cx}{1260x} \\ 3d\times\frac{1260x}{7x} & = 3d\times 180 & = & 540d & \frac{3d}{7x} = \frac{540d}{1260x} \\ 4e\times\frac{1260x}{21x} & = 4e\times 60 & = & 240e & \frac{4e}{21x} = \frac{240e}{1260x} \\ 5h\times\frac{1260x}{63x} & = 5h\times 20 & = & 100h & \frac{5h}{63x} = \frac{100h}{1260x}, \end{array}$$

which are the equivalent fractions having a common denominator.

3. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$ and d, to equivalent fractions having a common denominator.

Ans. $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, and $\frac{6acd}{6ac}$.

4. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a + \frac{4x}{5}$, to fractions having a common denominator.

Ans. $\frac{45}{60}$, $\frac{40x}{60}$, and $\frac{60a + 48x}{60}$.

5. Reduce $\frac{a}{2}$, $\frac{3x}{7}$, and $\frac{a + x}{a - x}$, to fractions having a common denominator.

Ans. $\frac{7a^2 - 7ax}{14a - 14x}$, $\frac{6ax - 6x^2}{14a - 14x}$, and $\frac{14a + 14x}{14a - 14x}$.

6. Reduce $\frac{p}{xy}$, $\frac{q}{xz}$ and $\frac{r}{yz}$ to fractions having a common denominator.

Ans. $\frac{pz}{xyz}$, $\frac{qy}{xyz}$ and $\frac{rx}{xyz}$.

7. Reduce $\frac{1}{a^2b^3}$, $\frac{1}{a^3b^2}$ and $\frac{1}{ab^4}$ to fractions having a common denominator.

Ans. $\frac{ab}{a^3b^4}$, $\frac{b^2}{a^3b^4}$ and $\frac{a^2}{a^3b^4}$.

8. Reduce $\frac{a}{4(a + x)}$, $\frac{b}{4(a - x)}$ and $\frac{c}{2(a^2-x^2)}$ to fractions having a common denominator.

Ans. $\frac{a^2 - ax}{4(a^2 - x^2)}$, $\frac{ab + bx}{4(a^2 - x^2)}$ and $\frac{2c}{4(a^2 - x^2)}$.

9. Reduce $\frac{p}{(a - x)^{\frac{1}{2}}}$, $\frac{q}{(a - x)^{\frac{3}{2}}}$ and $\frac{r}{(a - x)^{\frac{5}{2}}}$ to fractions having a common denominator.

Ans. $\frac{p(a - x)^2}{(a - x)^{\frac{5}{2}}}$, $\frac{q(a - x)}{(a - x)^{\frac{5}{2}}}$ and $\frac{r}{(a - x)^{\frac{5}{2}}}$.

CASE VII.

To add fractional quantities together.

Rule.—Reduce the fractions, if necessary, to a common denominator, then add all the numerators together, and their sum written above the common denominator will express the sum of the fractions.

The process will sometimes be much simplified by first finding the sum of two of the fractions having the simplest denominators, and then adding the remaining fraction or fractions to this sum.*

EXAMPLES.

1. Add $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$ together.

Here the least common denominator is 12 ; therefore

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=\frac{6x}{12}+\frac{4x}{12}+\frac{3x}{12}=\frac{13x}{12}=x+\frac{x}{12}.$$

2. Add $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ together.

$$\frac{a}{b}+\frac{c}{d}+\frac{e}{f}=\frac{a\cdot df}{b\cdot df}+\frac{c\cdot bf}{d\cdot bf}+\frac{e\cdot bd}{f\cdot bd}=\frac{adf+bcf+bde}{bdf}.$$

3. Add $a-\frac{3x^2}{b}$ and $b+\frac{2ax}{c}$ together.

$$a-\frac{3x^2}{b} \text{ and } +b+\frac{2ax}{c}=a+b+\frac{2ax}{c}-\frac{3x^2}{b}$$

$$=a+b+\frac{2abx-3cx^2}{bc}.$$

4. Add $\frac{2x}{5}$ and $\frac{5x}{7}$. Ans. $\frac{39x}{35}$.

---

* In adding or subtracting mixed quantities, it is best to bring the fractional parts only to a common denominator, and then to affix their sum or difference to the sum or difference of the integral parts, interposing the proper sign.

5. Add $\frac{3x}{2a}$ and $\frac{x}{5}$. Ans. $\frac{15x + 2ax}{10a}$.

6. Add $\frac{4x}{7}$ and $\frac{x - 2}{5}$. Ans. $\frac{27x - 14}{35}$.

7. Add $2a$, $3a + \frac{2x}{5}$ and $a - \frac{8x}{9}$. Ans. $6a - \frac{22x}{45}$.

8. Add $2a + \frac{3x}{5}$, $\frac{a}{a - x}$ and $\frac{a - x}{a}$.

Ans. $2a + \frac{10a(a - x) + 3ax(a - x) + 5x^2}{5a(a - x)}$.

9. Add $5x + \frac{x - 2}{3}$ and $4x - \frac{2x - 3}{5x}$.

Ans. $9x + \frac{5x^2 - 16x + 9}{15x}$.

10. Add $5x, \frac{2a}{3x^2}$ and $\frac{a + 2x}{4x}$. Ans. $5x + \frac{6x^2 + 3ax + 8a}{12x^2}$.

11. Add $\frac{2x}{1 - x^2}$ and $\frac{1}{x + 1}$. Ans. $\frac{1}{1 - x}$.

12. Add $\frac{1}{1 + a}$, $\frac{a}{1 + a}$, and $\frac{a}{1 - a}$. Ans. $\frac{1}{1 - a}$.

13. Add $\frac{b}{b + x}$, $\frac{bx}{(b + x)^2}$, and $\frac{x^2}{(b + x)^2}$. Ans. 1.

14. Add $\frac{p - q}{pq}$, $\frac{q - r}{qr}$ and $\frac{r - p}{pr}$. Ans. 0.

15. Add $\frac{2}{x - 1}$, $\frac{3}{x - 2}$ and $\frac{4}{x - 3}$. Ans. $\frac{9x^2 - 34x + 29}{x^3 - 6x^2 + 11x - 6}$.

16. Add $\frac{x^2 - xy + y^2}{x^2 + xy + y^2}$ and $\frac{x^2 + xy + y^2}{x^2 - xy + y^2}$.

Ans. $\frac{2(x^4 + 3x^2y^2 + y^4)}{x^4 + x^2y^2 + y^4}$.

CASE VIII.

To subtract one fractional quantity from another.

Rule.—Reduce the fractions to a common denominator, if necessary; subtract the requisite numerator from the other, under the difference write the common denominator, and it will give the difference of the fractions required.

EXAMPLES.

1. It is required to find the difference of $\frac{2x}{3}$ and $\frac{3x}{5}$.

$$\left.\begin{aligned} \text{Here } 2x \times 5 &= 10x \\ 3x \times 3 &= 9x \end{aligned}\right\} \text{the numerators.}$$

And $3 \times 5 = 15$, the common denominator.

Whence $\frac{10x}{15} - \frac{9x}{15} = \frac{x}{15}$, the difference required.

2. It is required to find the difference of $\frac{x - a}{2b}$ and $\frac{2a - 4x}{3c}$.

$$\left.\begin{aligned} (x - a) \times 3c &= 3cx - 3ac \\ (2a - 4x) \times 2b &= 4ab - 8bx \end{aligned}\right\} \text{the numerators.}$$

And $2b \times 3c = 6bc$, the common denominator.

Whence $\frac{3cx - 3ac}{6bc} - \frac{4ab - 8bx}{6bc} = \frac{3cx - 3ac - 4ab + 8bx}{6bc}$, the difference required.

3. Required the difference of $\frac{12x}{7}$ and $\frac{3x}{5}$. Ans. $\frac{39x}{35}$.

4. Required the difference of $15y$ and $\frac{1 + 2y}{8}$. Ans. $\frac{118y - 1}{8}$.

5. Required the difference of $\frac{ax}{b - c}$ and $\frac{ax}{b + c}$. Ans. $\frac{2acx}{b^2 - c^2}$.

6. Required the difference of $x - \frac{x-a}{c}$ and $x + \frac{x}{2b}$.

Ans. $\frac{cx + 2bx - 2ab}{2bc}$.

7. Required the difference of $a + \frac{a-x}{a+x}$ and $a - \frac{a+x}{a-x}$.

Ans. $\frac{2a^2 + 2x^2}{a^2 - x^2}$.

8. It is required to find the difference of $ax + \frac{2x+7}{8}$ and $x - \frac{5x-6}{21}$.

Ans. $\frac{168ax - 86x + 99}{168}$.

9. It is required to find the difference of $2x + \frac{3x-5}{7}$ and $3x + \frac{11x-10}{15}$.

Ans. $\frac{137x + 5}{105}$.

10. It is required to find the difference of $a + \frac{a-x}{a(a+x)}$ and $\frac{a+x}{a(a-x)}$.

Ans. $a - \frac{4x}{a^2 - x^2}$.

11. Find the excess of $\frac{a+2x}{a-2x}$ above $\frac{a-2x}{a+2x}$, and of $\frac{1}{x-6}$ above $\frac{1}{x-4}$.

Ans. $\frac{8ax}{a^2 - 4x^2}$ and $\frac{2}{x^2 - 10x + 24}$.

12. From the sum of $\frac{x+a}{x-a}$ and $\frac{x-a}{x+a}$ subtract $\frac{2(x-a)}{x+a}$.

Ans. $\frac{4ax}{x^2 - a^2}$.

13. From $\frac{a}{(a-b)(x+a)}$ subtract $\frac{b}{(a-b)(x+b)}$.

Ans. $\frac{x}{(x+a)(x+b)}$.

CASE IX.

To multiply fractional quantities together.

Rule.—Multiply the numerators together for a new numerator, and the denominators for a new denominator; and the former of these, being placed over the latter, will give the product of the fractions, as required.*

EXAMPLES.

1. It is required to find the product of $\frac{x}{6}$ and $\frac{2x}{9}$.

Here $\frac{x \times 2x}{6 \times 9} = \frac{2x^2}{54} = \frac{x^2}{27}$, the product required.

2. It is required to find the continued product of $\frac{x}{2}$, $\frac{4x}{5}$, and $\frac{10x}{21}$.

Here $\frac{x \times 4x \times 10x}{2 \times 5 \times 21} = \frac{40x^3}{210} = \frac{4x^3}{21}$, the product required.

3. It is required to find the product of $\frac{x}{a}$ and $\frac{a + x}{a - x}$.

Here $\frac{x \times (a + x)}{a \times (a - x)} = \frac{x^2 + ax}{a^2 - ax}$ the product required.

4. Required the product of $\frac{3x}{2}$ and $\frac{5x}{3b}$. Ans. $\frac{5x^2}{2b}$.

5. Required the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$. Ans. $\frac{3x^3}{5a}$.

---

* When the numerator of one of the fractions to be multiplied, and the denominator of the other, can be divided by some quantity which is common to each of them, the quotients may be used instead of the fractions themselves.

Also, when a fraction is to be multiplied by an integer, it is the same whether the numerator be multiplied by it, or the denominator divided by it.

Or if an integer is to be multiplied by a fraction, or a fraction by an integer, the integer may be considered as having unity for its denominator, and the two may be then multiplied together in the usual manner.

6. It is required to find the continued product of $\frac{2x}{3}$, $\frac{4x^2}{7}$ and $\frac{a}{a+x}$.

Ans. $\frac{8ax^3}{21a+21x}$.

7. It is required to find the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{5ac}{2b}$.

Ans. $15ax$.

8. It is required to find the product of $2a + \frac{bx}{a}$ and $6a - \frac{b}{ax}$.

Ans. $12a^2 + 6bx - \frac{2b}{x} - \frac{b^2}{a^2}$.

9. It is required to find the continued product of $3x$, $\frac{x+1}{2a}$ and $\frac{x-1}{a+b}$.

Ans. $\frac{3x^3 - 3x}{2a^2 + 2ab}$.

10. It is required to find the continued product of $\frac{a^2 - x^2}{a+b}$, $\frac{a^2 - b^2}{ax + x^2}$ and $a + \frac{ax}{a-x}$.

Ans. $\frac{a^3 - a^2b}{x}$.

11. Find the product of $x + \frac{2xy}{x-y}$ and $x - \frac{2xy}{x+y}$. Ans. x^2.

12. Find the product of $a^2 + ab + b^2$ and $\frac{1}{a^2} - \frac{1}{ab} + \frac{1}{b^2}$.

Ans. $\frac{a^2}{b^2} + \frac{b^2}{a^2} + 1$, or $\frac{a^4 + a^2b^2 + b^4}{a^2b^2}$.

13. Find the product of $\frac{a^3 - x^3}{a^3 + x^3}$ and $\frac{(a+x)^2}{(a-x)^2}$.

Ans. $\frac{a^3 + 2a^2x + 2ax^2 + x^3}{a^3 - 2a^2x + 2ax^2 - x^3}$.

14. Find the product of $\frac{x^2 + 3x + 2}{x^2 + 2x + 1}$ and $\frac{x^2 + 5x + 4}{x^2 + 7x + 12}$.

Ans. $\frac{x+2}{x+3}$

CASE X.

To divide one fractional quantity by another.

Rule.—Multiply the denominator of the divisor by the numerator of the dividend, for the numerator; and the numerator of the divisor by the denominator of the dividend, for the denominator. Or, which is more convenient in practice, multiply the dividend by the reciprocal of the divisor, and the product will be the quotient required.*

EXAMPLES.

1. It is required to divide $\frac{x}{3}$ by $\frac{2x}{9}$.

$$\text{Here } \frac{x}{3} \div \frac{2x}{9} = \frac{x}{3} \times \frac{9}{2x} = \frac{9x}{6x} = \frac{3}{2} = 1\tfrac{1}{2} \text{ Ans.}$$

2. It is required to divide $\frac{2a}{b}$ by $\frac{4c}{d}$.

$$\text{Here } \frac{2a}{b} \times \frac{d}{4c} = \frac{2ad}{4bc} = \frac{ad}{2bc} \text{ Ans.}$$

3. It is required to divide $\frac{x+a}{x-b}$ by $\frac{x+b}{5x+a}$.

$$\text{Here } \frac{x+a}{x-b} \times \frac{5x+a}{x+b} = \frac{5x^2+6ax+a^2}{x^2-b^2} \text{ Ans.}$$

4. It is required to divide $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{x+a}$.

$$\text{Here } \frac{2x^2}{a^3+x^3} \times \frac{a+x}{x} = \frac{2x^2(a+x)}{x(a^3+x^3)} = \frac{2x^2(a+x)}{x(a^2-ax+x^2).(a+x)} = \frac{2x}{a^2-ax+x^2}.$$

---

* When a fraction is to be divided by an integer, it is the same whether the numerator be divided by it, or the denominator multiplied by it.

Also, when the two numerators, or the two denominators, can be divided by some common quantity, that quantity may be thrown out of each, and the quotients used instead of the fractions proposed.

5. It is required to divide $\frac{7x}{5}$ by $\frac{3}{x}$, and $\frac{4x^2}{7}$ by $5x$.

Ans. $\frac{7x^2}{15}$ and $\frac{4x}{35}$.

6. It is required to divide $\frac{x+1}{6}$ by $\frac{2x}{3}$, and $\frac{x}{1-x}$ by $\frac{x}{5}$.

Ans. $\frac{x+1}{4x}$ and $\frac{5}{1-x}$.

7. It is required to divide $\frac{2ax+x^2}{c^3-x^3}$ by $\frac{x}{c-x}$.

Ans. $\frac{2a+x}{c^2+cx+x^2}$.

8. It is required to divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$.

Ans. $\frac{x^2+b^2}{x}$.

9. Divide $x+y+\frac{x^2}{y}$ by $y+x+\frac{y^2}{x}$, and $\frac{x^2-9}{x^2+4x+4}$ by $\frac{x+3}{x+2}$.

Ans. $\frac{x}{y}$, and $\frac{x-3}{x+2}$.

10. Divide 1 by $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $\frac{x^2-9x+20}{x^2-8x+12}$ by $\frac{x^2-12x+35}{x^2-7x+6}$.

Ans. $\frac{abc}{ab+bc+ac}$, and $\frac{x^2-5x+4}{x^2-9x+14}$.

11. Divide $x^6+x^4+x^2+1+x^{-6}+x^{-4}+x^{-2}+1$ by $x^3+x^{-3}+x+x^{-1}$.

Ans. x^3+x^{-3} or $x^3+\frac{1}{x^3}$

EXAMPLES FOR PRACTICE ON FRACTIONS.

Simplify the following expressions :—

1. $\frac{a-1}{2}+\frac{a-2}{3}+\frac{a+7}{6}$. Ans. a.

2. $\left\{a+\frac{b-a}{1+ab}\right\}\div\left\{1-a.\frac{b-a}{1+ab}\right\}$. Ans. b.

3. $\frac{a+b}{b}-\frac{2a}{a+b}+\frac{a^2(a-b)}{b(b^2-a^2)}$. Ans. $\frac{b}{a+b}$.

4. $\frac{1}{a-1}-\frac{1}{2(a+1)}-\frac{a+3}{2(a^2+1)}$. Ans. $\frac{a+3}{a^4-1}$.

5. $\frac{1}{x^3}+\frac{1}{x^2}-\frac{1}{x}-\frac{1}{(x^2+1)^2}+\frac{x-1}{x^2+1}$. Ans. $\frac{x^2+x+1}{x^3(x^2+1)^2}$.

6. $\frac{a+b+c}{2(x-1)^3}+\frac{3a+b-c}{2^2(x-1)^2}+\frac{a-b+c}{2^3(x-1)}-\frac{a-b+c}{2^3(x+1)}$.

Ans. $\frac{ax^2+bx+c}{(x-1)^2(x^2-1)}$.

7. $\dfrac{v}{1+\dfrac{v}{1+v+\dfrac{v}{1+v+v^2}}}$. Ans. $\frac{v^4+2v^3+3v^2+v}{2v^3+3v^2+4v+1}$.

8. $\frac{1}{(a-b)(a-c)(x+a)}+\frac{1}{(b-a)(b-c)(x+b)}+\frac{1}{(c-a)(c-b)(x+c)}$.

Ans. $\frac{1}{(x+a)(x+b)(x+c)}$.

9. $\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-a)(b-c)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}$.

Ans. $\frac{x}{(x-a)(x-b)(x-c)}$.

10 $\frac{a^2}{(a-b)(a-c)(x-a)}+\frac{b^2}{(b-a)(b-c)(x-b)}+\frac{c^2}{(c-a)(c-b)(x-c)}$.

Ans. $\frac{x^2}{(x-a)(x-b)(x-c)}$.

11. $\frac{a^2+pa+q}{(a-b)(a-c)(x-a)}+\frac{b^2+pb+q}{(b-a)(b-c)(x-b)}+\frac{c^2+pc+q}{(c-a)(c-b)(x-c)}$.

Ans. $\frac{x^2+px+q}{(x-a)(x-b)(x-c)}$.

12. $\frac{1}{(a-b)(x+b)}+\frac{1}{(b-a)(x+a)}$. Ans. $\frac{1}{(x+a)(x+b)}$.

13. $\dfrac{1}{(a-b)(a-c)(x-a)}+\dfrac{1}{(b-a)(b-c)(x-b)}+\dfrac{1}{(c-a)(c-b)(x-c)}$.

Ans. $\dfrac{1}{(x-a)(x-b)(x-c)}$.

14. $\dfrac{x^2-9x+20}{x^2-6x}\div\dfrac{x^2-5x}{x^2-13x+42}$. Ans. $\dfrac{(x-4)(x-7)}{x^2}$.

15. $\dfrac{1}{\dfrac{1}{x}+\dfrac{1}{x+1}+\dfrac{1}{x+2}}-\dfrac{1}{\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{1}{x-2}}$.

Ans. $\dfrac{6x^2(x^2-2)}{9x^4-24x^2+4}$.

16. $\dfrac{x^{3n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}+\dfrac{1}{x^n+1}-\dfrac{1}{x^n-1}$. Ans. $x^{2n}+2$.

INVOLUTION.

Involution is the raising of powers from any proposed root; or the method of finding the square, cube, biquadrate, or fourth power, &c., of any given quantity.

Rule. I. Multiply the index of the quantity by the index of the power to which it is to be raised, and the result will be the index of the power required.

2. Or multiply the quantity into itself as many times less one as is denoted by the index of the power, and the last product will be the answer.

Note. When the sign of the root is +, all the powers of it will be +; and when the sign is −, all the even powers will be +, and the odd powers −; as is evident from multiplication.*

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* Any power of the product of two or more quantities is equal to the same power of each of the factors multiplied together. And any power of a fraction is equal to the same power of the numerator divided by the like power of the denominator, as may be easily shewn.

Also, a^m raised to the nth power is $a^m\times a^m\times a^m\ldots\ldots$ to n terms, or a^{mn} and $-a^m$ raised to the nth power is $\pm a^{mn}$, according as the index n is an even or an odd number.

EXAMPLES.

a the root.
$a^2 =$ square.
$a^3 =$ cube.
$a^4 =$ 4th power.
$a^5 =$ 5th power.
&c.

a^2 the root.
$a^4 =$ square.
$a^6 =$ cube.
$a^8 =$ 4th power
$a^{10} =$ 5th power.
&c.

$-\ 3a$ the root.
$+\ 9a^2 =$ square.
$-27a^3 =$ cube.
$+81a^4 =$ 4th power.
&c.

$-\ 2ax^2$ the root.
$+\ 4a^2x^4 =$ square.
$-\ 8a^3x^6 =$ cube.
$+16a^4x^8 =$ 4th power.
&c.

$\frac{x}{a}$ the root.
$\frac{x^2}{a^2} =$ square.
$\frac{x^3}{a^3} =$ cube.
$\frac{x^4}{a^4} =$ 4th power.
&c.

$-\frac{2ax^2}{3b}$ the root.
$+\frac{4a^2x^4}{9b^2} =$ square.
$-\frac{8a^3x^6}{27b^3} =$ cube.
$+\frac{16a^4x^8}{81b^4} =$ 4th power.
&c.

$$\begin{array}{l}
a + x \text{ .... root.}\\
\underline{a + x}\\
a^2 + ax\\
\underline{\quad + ax + x^2}\\
a^2 + 2ax + x^2 \text{ .... square.}\\
\underline{a \ + x}\\
a^3 + 2a^2x + ax^2\\
\underline{\quad + \ a^2x + 2ax^2 + x^3}\\
a^3 + 3a^2x + 3ax^2 + x^3 \text{... cube.}
\end{array}$$

$$\begin{array}{l}
a - x \text{ .... root.}\\
\underline{a - x}\\
a^2 - ax\\
\underline{\quad - ax \ + x^2}\\
a^2 - 2ax + x^2 \text{ ...... square.}\\
\underline{a \ - x}\\
a^3 - 2a^2x + \ ax^2\\
\underline{\quad - \ a^2x + 2ax^2 - x^3}\\
a^3 - 3a^2x + 3ax^2 - x^3 \text{.. cube.}
\end{array}$$

In like manner,

$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$ .... 4th power.
$(a-x)^5 = a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$ .... 5th power.
$(a+x)^6 = a^6 + 6a^5x + 15d^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$ ... 6th power.

EXAMPLES FOR PRACTICE.

1. Required the third power of $2a^2$. Ans. $8a^6$.

2. Required the 4th power of $-2a^2x$. Ans. $16a^8x^4$.

3. Required the third power of $-\frac{2}{3}x^2y^3$. Ans. $-\frac{8}{27}x^6y^9$.

4. Required the 4th power of $-\frac{3a^2x}{5b^2}$. Ans. $\frac{81a^8x^4}{625b^8}$.

5. Find the square of $x + x^{-1}$ and the cube of $x^2 + 1 + x^{-2}$.
Ans. x^2+2+x^{-2} and $x^6+3x^4+6x^2+7+6x^{-2}+3x^{-4}+x^{-6}$.

6. Find the cube of $a^m + 1$, and the mth power of $2a^2x^{3n}$.
Ans. $a^{3m} + 3a^{2m} + 3a^m + 1$ and $2^m a^{2m} x^{3mn}$.

By considering attentively the laws followed in the preceding examples, there may readily, by induction, be obtained the following

Rule. II. A binomial or residual quantity may also be raised to any power, without the trouble of continual involution, as follows :—

1. Find the terms without the coefficients, by observing that the index of the first, or leading quantity, begins with that of the given power, and decreases continually by 1, in every term to the last; and that in the following quantity, the indices of the terms are 1, 2, 3, 4, &c.

2. To find the coefficients, observe that those of the first and last terms are always 1 ; and that the coefficient of the second term is the index of the power of the first: and for the rest, if the coefficient of any term be multiplied by the index of the leading quantity in it, and the product be divided by the number of terms to that place, it will give the coefficient of the term next following; or instead of dividing the product of the coefficient and exponent, divide only one of them by the number, whenever such division is practicable, and multiply the quotient by the other. It may be observed that the coefficient of each succeeding term is equal to the product of the coefficient of the preceding term by the index of the leading quantity, divided by the index of the following quantity increased by unity. In this way the general result may be put down at sight.

Note.—The whole number of terms will be one more than the index of the given power; and when both terms of the root are +, all the terms of the power will be + ; but if the second term be −, all the odd terms will be +, and the even terms − ; or, which is the same thing, the terms will be + and − alternately, the coefficients equidistant from each extreme are equal, while the sum of the indices will be equal, in each term, to the index of the power required.

Rule III.—For the purpose of generalisation, suppose it were required to raise the binomial $a \pm b$ to any power denoted by n. Proceeding with n as has been done with the different indices in the preceding examples, we shall have

$$(a+b)^n = a^n + na^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 + \frac{n\,(n-1)}{2}.\frac{n-2}{3}\,a^{n-3}b^3$$

$$+\frac{n(n-1)\,(n-2)}{2\;.\;3}.\,\frac{n-3}{4}\,a^{n-4}b^4 + \ldots + b^n \ldots . (1)$$

Writing $-b$ for b in the formula (1), we have

$$(a-b)^n = a^n - na^{n-1}b + n.\,\frac{n-1}{2}a^{n-2}b^2 - \frac{n\,(n-1)}{2}.\frac{n-2}{3}a^{n-3}b^3$$

$$+\frac{n(n-1)\,(n-2)}{2\;.\;3}.\frac{n-3}{4}\,a^{n-4}b^4 - \ldots \pm b^n \ldots . (2),$$

forming what is known by the name of the binomial theorem of Newton, which, as will be afterwards shewn, holds true whether n be a whole number or a fraction, and either positive or negative, though, if a fraction or negative, the series will not terminate.

From the great utility of this theorem in all branches of pure mathematics, and their practical applications, it should be well recollected and readily applied by every student.

By a little consideration of the several terms of the formula (1), it will be seen that the pth term is

$$\frac{n(n-1)\,(n-2)\ldots\ldots\{n-(p-2)\}}{1\;.\;2\;.\;3\;\ldots\ldots\;(p-1)}a^{n-(p-1)}b^{p-1}\;\ldots . \qquad (3)$$

And from this general expression, any term of the formula (1) may readily be obtained. Thus, if the *fourth* term were required, we should have $p = 4$, and the last factor in the numerator of the fractional part of (3), viz., $n - (p - 2)$, would become $n - 2$, while the last factor in the denominator would be 3 ; therefore the fourth term is

$$\frac{n(n-1)\,(n-2)}{1\;.\;2\;.\;3}a^{n-3}b^3.$$

Note. 1. The index of the first or leading quantity, a, in the formula (1) is the index of the given power, and decreases continually by unity in every term to the last. The index of the other quantity, b, increases continually, by unity, from 1 in the *second* term to n in the *last* term, and therefore we learn that the whole number of terms in any integral power of a binomial or residual expression is one more than the index of the proposed power.

Note. 2. The coefficient of any term in the formula (1) may be found from that of the preceding term, by multiplying the coefficient of the preceding term by the index of a in that term, and dividing the product by the *number* of the same term, reckoning from the first term, or simply by the *index* of the second quantity of the preceding term increased by *unity*. Thus, to find the coefficient of the fourth term, multiply $\frac{n(n-1)}{2}$, the coefficient of the *third* term, by $n-2$, the index of a in that term, and divide the product by 3, the number of the term, or the index of b in the third term, increased by 1,—the quotient will be $\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}$, the coefficient of the fourth term, and so on. This mode of finding the coefficients of the successive terms is of great use in the practical application of the theorem, as will appear in the following

EXAMPLES.

1. Find the fifth power of $a+x$, or expand $(a+x)^5$.

Comparing this with $(a+b)^n$ in the formula (1), we have $b=x$, and $n=5$: hence

$$(a+x)^5 = a^5 + 5a^4x + 5.\,\frac{4}{2}a^3x^2 + 10.\,\frac{3}{3}a^2x^3 + \frac{10\times 2}{4}ax^4 + \frac{5\times 1}{5}x^5$$

$$= a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.$$

2. Find the fifth term of $(2+x)^7$.

By formula (1), $a=2$, $b=x$, $n=7$, and since $p=5$, the expression (3) gives the fifth term of $(2+x)^7 =$

$$\frac{7\,.\,6\,.\,5\,.\,4}{1\,.\,2\,.\,3\,.\,4}\,2^3x^4 = 35\times 8x^4 = 280x^4.$$

3. Find the fourth power of $a-2x$ and the cube of $3a-2y$.

Ans. $(a-2x)^4 = a^4 - 8a^3x + 24a^2x^2 - 32ax^3 + 16x^4.$
$(3a-2y)^3 = 27a^3 - 54a^2y + 36ay^2 - 8y^3.$

4. Find the third term of $(2ax+3x^2)^5$; and the fourth term of $(a^2+x^2)^6$. Ans. $720a^3x^7$ and $20a^6x^6$.

5. Find the cube of $a - bx + cx^2$, and thence deduce the cube of $1 - 2x + 3x^2$.

Ans. $(a-bx+cx^2)^3 = a^3 - 3a^2bx + 3a(b^2+ac)x^2 - b(b^2+6ac)x^3 + 3c(b^2+ac)x^4 - 3bc^2x^5 + c^3x^6$,

and $(1-2x+3x^2)^3 = 1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.

6. Find the fourth power of $mx - ny$, and the cube of $e^x - e^{-x}$.

Ans. $(mx-ny)^4 = m^4x^4 - 4m^3nx^3y + 6m^2n^2x^2y^2 - 4mn^3xy^3 + n^4y^4$,

and $(e^x - e^{-x})^3 = e^{3x} - 3e^x + 3e^{-x} - e^{-3x}$.

7. Shew that,

$$(a+x)^3 + (a-x)^3 = 2a^3 + 6ax^2$$
$$(a+x)^3 - (a-x)^3 = 6a^2x + 2x^3.$$

8. Shew that the sum of the coefficients of the terms of $(a+b)^n$ is $= 2^n$; and the sum of those of $(a-b)^n$ is $= 0$.

9. Shew that the sum of the coefficients of the even terms of $(a \pm b)^n$ is equal to the sum of the coefficients of the odd terms, though in the second case with contrary signs.

To find the square of a polynomial quantity.

Since

$$\begin{aligned}(a+b+c+d)^2 &= \{(a+b)+(c+d)\}^2 = (a+b)^2 + 2(a+b)(c+d) + (c+d)^2 \\ &= a^2 + 2ab + b^2 + 2ac + 2ad + 2bc + 2bd + c^2 + 2cd + d^2 \\ &= a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd):\end{aligned}$$

therefore we have the following rule for finding at once the square of a polynomial quantity.

Rule.—To the sum of the squares of each of the terms of the polynomial quantity add twice the sum of the products of the terms taken two and two.

EXAMPLES.

1. Find the square of $1 + x + x^2 + x^3$.

$$\begin{aligned}(1+x+x^2+x^3)^2 &= 1 + x^2 + x^4 + x^6 + 2(x + x^2 + x^3 + x^3 + x^4 + x^5) \\ &= 1 + 2x + 3x^2 + 4x^3 + 3x^4 + 2x^5 + x^6.\end{aligned}$$

2. What is the square of $x - y + 2z - 3v$ equal to?

Ans. $x^2 + y^2 + 4z^2 + 9v^2 - 2xy + 4xz - 6xv - 4yz + 6yv - 12zv$.

3. Find the square of $a^2 - x^2 + 2ax + b^2$.

Ans. $a^4 + 2a^2b^2 + b^4 + 4a(a^2+b^2)x + 2(a^2-b^2)x^2 - 4ax^3 + x^4$.

EVOLUTION.

Evolution, or the extraction of roots, is the method of finding the square root, the cube root, &c., of any given quantity, and is hence the inverse of involution.

CASE I.

To find any root of a simple or monomial quantity.

Rule.—Extract the root of each factor separately, and the product of these roots will be the root required. Thus,

$$\sqrt{9a^2x^2} = \sqrt{9} \times \sqrt{a^2} \times \sqrt{x^2} = 3ax.$$

If the proposed quantity is fractional, divide the required root of the numerator by the same root of the denominator. Thus,

$$\sqrt[3]{\frac{8x^3}{27y^6}} = \frac{\sqrt[3]{8x^3}}{\overline{27y^6}} = \frac{\sqrt[3]{8} \times \sqrt[3]{x^3}}{\sqrt{27} \times \sqrt[3]{y^6}} = \frac{2x}{3y^2}.$$

Note.—The square root, the fourth root, or any other even root, of an affirmative quantity, may be either + or –. Thus, $\sqrt{a^2} = +a$ or $-a$, and $\sqrt[4]{b^4} = +b^2$ or $-b^2$, &c. But the cube root, or any other odd root, of a quantity, will have the same sign as the quantity itself. Thus,

$$\sqrt[3]{a^3} = a\,;\ \sqrt[3]{-a^3} = -a\,;\ \sqrt[5]{-a^5} = -a, \text{ \&c.*}$$

It may here, also, be further remarked, that any even root of a negative quantity is unassignable.

---

* The reason why $+a$ and $-a$ are each the square root of a^2 is obvious, since, by the rule of multiplication, $(+a) \times (+a)$ and $(-a) \times (-a)$ are both equal to a^2.

And for the cube root, fifth root, &c., of a negative quantity, it is plain, from the same rule, that

$$(-a) \times (-a) \times (-a) = -a^3;\ \text{and}\ (-a^3) \times (+a^2) = -a^5.$$

And consequently $\sqrt[3]{-a^3} = -a$, and $\sqrt[5]{-a^5} = -a$.

Thus, $\sqrt{-a^2}$ cannot be determined, as there is no quantity, either positive or negative (+ or −), that, when multiplied by itself, will produce $-a^2$.

EXAMPLES.

1. Find the square root of $9x^2$; and the cube root of $8x^3$.

$$\text{Here } \sqrt{9x^2} = \sqrt{9} \times \sqrt{x^2} = 3 \times x = 3x. \quad \text{Ans.}$$
$$\text{And } \sqrt[3]{8x^3} = \sqrt[3]{8} \times \sqrt[3]{x^3} = 2 \times x = 2x. \quad \text{Ans.}$$

2. It is required to find the square root of $\frac{a^2x^2}{4c^2}$, and the cube root of $-\frac{8a^3x^3}{27c^3}$.

$$\text{Here } \sqrt{\frac{a^2x^2}{4c^2}} = \frac{\sqrt{a^2x^2}}{\sqrt{4c^2}} = \frac{ax}{2c}; \text{ and } \sqrt[3]{-\frac{8a^3x^3}{27c^3}} = -\frac{2ax}{3c}.$$

3. Find the square root of $4a^2x^6$, and the cube root of $-125a^3x^6$.
Ans. $2ax^3$ and $-5ax^2$.

4. Find the 4th root of $256a^4x^8$, and the square root of $\frac{4a^4}{9x^2y^2}$.
Ans. $4ax^2$ and $\frac{2a^2}{3xy}$.

5. Find the cube root of $\frac{8a^3}{125x^6}$, and the 5th root of $-\frac{32a^5x^{10}}{243}$.
Ans. $\frac{2a}{5x^2}$ and $\frac{-2ax^2}{3}$.

6. Find the square roots of $a^{2m}x^{4n}$ and $64a^4x^2$, and the nth root of $a^{mn}x^{pn-n}$.
Ans. $a^m x^{2n}$, $8a^2x$, and $a^m x^{p-1}$.

7. Find the nth root of $2^n a^{2n} x^{pmn}$, and the pth root of $\frac{h^p x^{np} y^{2pq}}{a^{3p} z^{3pq}}$.
Ans. $2a^2x^{pm}$, and $\frac{hx^n y^{2q}}{a^3 z^{3q}}$.

CASE II.

To extract the square root of a compound quantity.

Rule. 1.—Range the terms, of which the quantity is composed, according to the dimensions of some letter in them, beginning with the highest, and place the root of the first term in the quotient.

2. Subtract the square of the root, thus found, from the first term, and bring down the next two terms to the remainder, for a dividend.

3. Divide this dividend by double that part of the root already determined, and place the result both in the quotient and divisor.

4. Multiply the divisor, so increased, by the term of the root last placed in the quotient, and subtract the product from the dividend; and so on, as in common arithmetic.

EXAMPLES.

1. Extract the square root of $x^4 - 4x^3 + 6x^2 - 4x + 1$.

$$\begin{array}{rl}
 & x^4 - 4x^3 + 6x^2 - 4x + 1\ (x^2 - 2x + 1 = \text{root required} \\
 & x^4 \\
\hline
2x^2 - 2x) & -4x^3 + 6x^2 \\
 & -4x^3 + 4x^2 \\
\hline
2x^2 - 4x + 1) & 2x^2 - 4x + 1 \\
 & 2x^2 - 4x + 1 \\
\hline
 & *
\end{array}$$

2. Extract the square root of $4a^4 + 12a^3x + 13a^2x^2 + 6ax^3 + x^4$.

$$\begin{array}{rl}
 & 4a^4 + 12a^3x + 13a^2x^2 + 6ax^3 + x^4\ (2a^2 + 3ax + x^2 = \text{root.} \\
 & 4a^4 \\
\hline
4a^2 + 3ax) & 12a^3x + 13a^2x^2 \\
 & 12a^3x + \ \ 9a^2x^2 \\
\hline
4a^2 + 6ax + x^2) & 4a^2x^2 + 6ax^3 + x^4 \\
 & 4a^2x^2 + 6ax^3 + x^4 \\
\hline
 & *
\end{array}$$

Note.—When the quantity to be extracted has no exact root, the operation may be carried on as far as is thought necessary, or till the regularity of the terms shews the law by which the series would be continued.

EXAMPLE.

1. It is required to extract the square root of $1 + x$.

$$
\begin{array}{r|l}
 & 1+x\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}, \text{ \&c.}\right. \\
 & 1 \\
\hline
2+\frac{x}{2} & x \\
 & x+\frac{x^2}{4} \\
\hline
2+x-\frac{x^2}{8} & -\frac{x^2}{4} \\
 & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
\hline
2+x-\frac{x^2}{4}+\frac{x^3}{16} & \frac{x^3}{8}-\frac{x^4}{64} \\
 & \frac{x^3}{8}+\frac{x^4}{16}-\frac{x^5}{64}+\frac{x^6}{256} \\
\hline
 & -\frac{5x^4}{64}+\frac{x^5}{64}-\frac{x^6}{256}.
\end{array}
$$

Here, if the numerators and denominators of the last two terms be each multiplied by 3, which will not alter their values, the root will become

$$1+\frac{x}{2}-\frac{x^2}{2.4}+\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}+\frac{3.5.7x^5}{2.4.6.8.10}-\ldots,$$

where the law of the series is manifest.

EXAMPLES FOR PRACTICE.

Find the square roots of the following quantities :—

	Answers.
1. $a^4+4a^3x+6a^2x^2+4ax^3+x^4$.	$a^2+2ax+x^2$.
2. $x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16}$.	$x^2-x+\frac{1}{4}$.
3. $4x^6-4x^4+12x^3+x^2-6x+9$.	$2x^3-x+3$.
4. $x^6+4x^5+10x^4+20x^3+25x^2+24x+16$.	x^3+2x^2+3x+4.
5. $4a^2b^4-12a^3b^3+17a^4b^2-12a^5b+4a^6$.	$2ab^2-3a^2b+2a^3$.

	Answers.
6. $x^4 - 2px^3 + (p^2+2q)x^2 - 2pqx + q^2$.	$x^2 - px + q$.
7. $x^{2n}y^{4n} - 2x^ny^{2n}z^{3n} + z^{6n}$.	$x^ny^{2n} - z^{3n}$.
8. $x^{2n} + x^{-2n} + 2$.	$x^n + x^{-n}$.
9. $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} + \frac{xy}{3} + \frac{yz}{6} + \frac{xz}{4}$.	$\frac{x}{2} + \frac{y}{3} + \frac{z}{4}$.
10. $\frac{a^4}{16} + \frac{3a^3}{2b} + \frac{12a^2}{b^2} + \frac{36a}{b^3} + \frac{36}{b^4}$.	$\frac{a^2}{4} + \frac{3a}{b} + \frac{6}{b^2}$.
11. $\frac{x^2}{y^2} + \frac{y^2}{x^2} - 2\left(\frac{x}{y} + \frac{y}{x}\right) + 3$.	$\frac{x}{y} + \frac{y}{x} - 1$.
12. $y^2 + \frac{2a}{3}y - by + \frac{a^2}{9} + \frac{b^2}{4} - \frac{ab}{3}$.	$y + \frac{a}{3} - \frac{b}{2}$.
13. $25\frac{3}{7} - \left(\frac{15}{2}\frac{y}{x} + \frac{20}{7}\frac{x}{y}\right) + \frac{4x^2}{49y^2} + \frac{9y^2}{16x^2}$.	$\frac{2x}{7y} - 5 + \frac{3y}{4x}$.

14. Find the square root of 2 or of 1 + 1, and thence deduce the square root of $a^2 + b$.

$$\text{Ans. } \sqrt{2} = 1 + \frac{1}{2} - \frac{1}{2.4} + \frac{3}{2.4.6} - \frac{3.5}{2.4.6.8} + \frac{3.5.7}{2.4.6.8.10} - \ldots\ldots$$

$$\sqrt{a^2 + b} = a + \frac{b}{2a} - \frac{b^2}{2.4a^3} + \frac{3b^3}{2.4.6a^5} - \frac{3.5b^4}{2.4.6.8a^7} + \ldots\ldots$$

15. Required the square root of $a - b^2$.

$$\text{Ans. } a^{\frac{1}{2}} - \frac{b^2}{2a^{\frac{1}{2}}} - \frac{b^4}{8a^{\frac{3}{2}}} - \frac{b^6}{16a^{\frac{5}{2}}} - \frac{5b^8}{128a^{\frac{7}{2}}} - \ldots\ldots$$

16. Extract the square roots of $\frac{a^4}{a^2 - x}$ and of $\frac{a + x}{a - x}$.

$$\text{Ans. } a + \frac{x}{2a} + \frac{1.3x^2}{2.4a^3} + \frac{3.5x^3}{2.4.6a^5} + \ldots\ldots$$

$$\text{and } 1 + \frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{2a^3} + \ldots\ldots$$

17. Find the square root of 1 − 1, and thence deduce the square root of $a^2 - x^2$.

$$\text{Ans. } \sqrt{1 - 1} = 1 - \frac{1}{2} - \frac{1}{2.4} - \frac{3}{2.4.6} - \frac{3.5}{2.4.6.8} - \ldots\ldots$$

$$\text{and } \sqrt{a^2 - x^2} = a - \frac{x^2}{2a} - \frac{x^4}{2.4a^3} - \frac{3x^6}{2.4.6a^5} - \frac{3.5x^8}{2.4.6.8a^7} - \ldots\ldots$$

CASE III.

To extract any root of a compound quantity.

Rule.—Arrange the terms so that the indices of the powers of the letter or letters may either decrease or increase regularly from the first term to the last; then the required root of the first term will be the first term of the root, and subtract the corresponding power of this term from the given quantity; divide the first term of the remainder by twice the first term of the root, for the square root; by three times the square of the first term of the root, for the cube root, and so on, and the quotient will be the next term of the root. Involve the binomial part of the root so obtained to the proper power, and subtract the result from the given quantity. If there be no remainder, the operation is completed, but if there is a remainder, divide the first term of it by the same divisor as before, which will give the next term of the root; and proceed as before till the operation either terminates, or a sufficient number of terms has been obtained.*

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* Direct rules for the extraction of any root are easily investigated, but they are of little practical importance, and in many instances these rules are superseded by a slight inspection of the given quantity whose root is to be extracted. Thus, if the cube root of $8x^6 + 48x^5 - 60x^4 - 80x^3 - 90x^2 + 108x - 27$ were required, the cube root of the first term, $8x^6$, is $2x^2$, and the cube root of the last term, -27, is -3; therefore the extreme terms of the root are $2x^2$ and -3. But we know from the formation of a complete cube that the second term divided by three times the square of the root of the first term will give the second term of the root; hence $48x^5$ divided by $3(2x^2)^2$ or $12x^4$ gives $4x$, which is the second term of the root; therefore the root is $2x^2 + 4x - 3$, if the given quantity admits of an exact cube root. The cube of this root produces the proposed expression, and therefore the root is truly determined. The term, $4x$, of the root might also be found by dividing the last term but one by three times the square of the root of the last term; thus, $108x$ divided by $3(-3)^2$ or 27 gives $4x$ for a quotient as before. A similar inspection would give any root of a complete power. The fourth root may be obtained by first extracting the square root, and then the square root of the former square root.

All even roots, that is, roots the denominator of whose index is an even number, may be found in a similar manner. Thus, let the index of the root be $\frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$, then first extract the square root and then the cube root. Let the index of the root be $\frac{1}{9} = \frac{1}{3} \times \frac{1}{3}$, then first extract the cube root of the original quantity, and next the cube root of the cube root already found, and so on.

EXAMPLES.

1. Required the square root of $x^2 + 2xy + y^2 + 2xz + 2yz + z^2$.

$$x^2 + 2xy + y^2 + 2xz + 2yz + z^2(x + y + z$$

$$x^2 \qquad\qquad = \text{root.}$$

$$2x\ )\ 2xy$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$2x\ )\ 2xz$$

$$(x + y + z)^2 = x^2 + 2xy + y^2 + 2xz + 2yz + z^2$$

2. Required the square root of $a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4$.

$$a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4(a^2 - ax + x^2$$

$$a^4 \qquad\qquad = \text{root.}$$

$$2a^2) - 2a^3x$$

$$(a^2 - ax)^2 = a^4 - 2a^3x + a^2x^2$$

$$2a^2)2a^2x^2 = \text{difference}$$

$$(a^2 - ax + x^2)^2 = a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4$$

3. Required the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.

$$x^6 + 6x^5 - 40x^3 + 96x - 64(x^2 + 2x - 4$$

$$x^6 \qquad\qquad = \text{root.}$$

$$3x^4)\ \ 6x^5$$

$$(x^2 + 2x)^3 = x^6 + 6x^5 + 12x^4 + 8x^3$$

$$3x^4) - 12x^4 = \text{difference}$$

$$(x^2 + 2x - 4)^3 = x^6 + 6x^5 - 40x^3 + 96x - 64$$

4. Required the square root of $4a^2 - 12ax + 9x^2$. Ans. $2a - 3x$.

5. Required the square root of $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$.
Ans. $a + b + c$.

6. Required the cube root of $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.
Ans. $x^2 - 2x + 1$.

7. Required the 4th root of $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$.
Ans. $2a - 3x$.

8. Required the 5th root of $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$.
Ans. $2x - 1$.

9. Required the 10th root of $x^{10} + 10ax^9 + 45a^2x^8 + 120a^3x^7 + 210a^4x^6 + 252a^5x^5 + 210a^6x^4 + 120a^7x^3 + 45a^8x^2 + 10a^9x + a^{10}$.
Ans. $x + a$.

IRRATIONAL QUANTITIES OR SURDS.

An *irrational quantity*, or a *surd*, is a quantity which has no exact root, and is usually expressed by means of the radical sign or by a fractional index. When a fractional index is employed, the numerator denotes the power to which the quantity is to be raised, and the denominator, the root of this power to be extracted; or the denominator denotes the root of the quantity to be extracted, and the numerator the power to which such root is to be raised.

Thus $\sqrt{2}$ or $2^{\frac{1}{2}}$ denotes the square root of 2; $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ denotes either the cube root of the square of a, or the square of the cube root of a; and generally $a^{\frac{m}{n}}$ means the nth root of the mth power of a, or the mth power of the nth root of a.

CASE I.

To represent a rational quantity in an irrational or surd form.

Since $a = a^{\frac{1}{1}} = a^{\frac{2}{2}} = a^{\frac{3}{3}} = \ldots\ldots a^{\frac{m}{m}}$, and that $a^{\frac{m}{m}} = a^{m \times \frac{1}{m}} = (a^m)^{\frac{1}{m}}$; hence we have the following

Rule.—Raise the quantity to the power corresponding to the number denoting the root of the surd, and to this power affix the required index or radical sign.

Note.—Since $\sqrt[n]{a^p} \times \sqrt[n]{b^q} = \sqrt[n]{(a^p \times b^q)}$; therefore a *mixed* surd, as $2\sqrt{2}$ may be expressed as an entire surd; for by the rule $2 = \sqrt{2^2}$, $\therefore 2\sqrt{2} = \sqrt{2^2} \times \sqrt{2} = \sqrt{2^3} = \sqrt{8}$.

EXAMPLES.

1. Represent 3 in the form of the square root, and $2\sqrt[3]{4}$ as an entire surd.

   Here $3 \times 3 = 3^2 = 9$; therefore $3 = \sqrt{9}$, also $2 \times 2 \times 2 = 2^3 = 8$, $\therefore 2 = \sqrt[3]{8}$, and $2\sqrt[3]{4} = \sqrt[3]{8} \times \sqrt[3]{4} = \sqrt[3]{32}$.

2. Express 5 in the form of the square root, and $2x^2$ in the form of the cube root.

   Ans. $\sqrt{25}$, and $\sqrt[3]{8x^6}$ or $(8x^6)^{\frac{1}{3}}$.

3. Express $-3x$ in the form of the cube root, and $-2x$ in the form of the fourth root. Ans. $\sqrt[3]{-27x^3}$, and $\sqrt[4]{16x^4}$.

4. Let a^2 be expressed in the form of the fifth root, and $a + x$, $\sqrt{a} + \sqrt{b}$, $\frac{\sqrt{a}}{2a}$ and $\frac{a}{b\sqrt{a}}$ in the form of the square root.

   Ans. $\sqrt[5]{a^{10}}$, $(a^2 + 2ax + x^2)^{\frac{1}{2}}$, $(a + 2\sqrt{ab} + b)^{\frac{1}{2}}$, $\sqrt{\frac{1}{4a}}$ and $\sqrt{\frac{a}{b^2}}$.

5 Let $5\sqrt{6}$, $3\sqrt{a}$, $\frac{1}{2}\sqrt[3]{4a}$, $\frac{1}{5}\sqrt{5a}$ and $\frac{2a}{3}\sqrt[3]{\frac{9}{4a^2}}$ be expressed in simple radical forms or entire surds.

   Ans. $\sqrt{150}$, $\sqrt{9a}$, $\sqrt[3]{\frac{a}{2}}$, $\sqrt{\frac{a}{5}}$ and $\sqrt[3]{\frac{2a}{3}}$.

6. Express $(a + x)\left\{\frac{a - x}{a + x}\right\}^{\frac{1}{2}}$ and $\frac{x + 1}{x - 1}\left\{\frac{x - 1}{x + 1}\right\}^{\frac{1}{2}}$ in simple radical forms.

   Ans. $\sqrt{a^2 - x^2}$ and $\left(\frac{x + 1}{x - 1}\right)^{\frac{1}{2}}$.

CASE II.

To reduce quantities having different indices to others that shall have either a given or a common index.

Rule.—For a given index.—Divide the indices of the proposed quantities by the given index, and the quotients will be the new indices of those quantities; then write the quantities with their new indices under the proper radical sign or given index.

For a common index.—Reduce the indices of the quantities to a common denominator, and involve each to the power denoted by its numerator.

EXAMPLES.

1. Reduce $3^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$ to quantities that shall have the index $\frac{1}{6}$, and also $a^{\frac{2}{3}}$ and $a^{\frac{1}{2}}$ to quantities having a common index.

Here $\frac{1}{2} \div \frac{1}{6} = \frac{1}{2} \times \frac{6}{1} = \frac{6}{2} = 3$, the index of the first quantity,

$\frac{1}{3} \div \frac{1}{6} = \frac{1}{3} \times \frac{6}{1} = \frac{6}{3} = 2$, the index of the second quantity;

therefore $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}} = (3^3)^{\frac{1}{6}}$ and $(2^2)^{\frac{1}{6}} = 27^{\frac{1}{6}}$ and $4^{\frac{1}{6}}$.

Again, $a^{\frac{2}{3}}$ and $a^{\frac{1}{2}} = a^{\frac{4}{6}}$ and $a^{\frac{3}{6}} = (a^4)^{\frac{1}{6}}$ and $(a^3)^{\frac{1}{6}}$.

2. Reduce $5^{\frac{1}{2}}$ and $6^{\frac{1}{3}}$ to quantities that shall have the common index $\frac{1}{6}$.

Ans. $125^{\frac{1}{6}}$ and $36^{\frac{1}{6}}$.

3. Let $2^{\frac{1}{2}}$ and $4^{\frac{1}{4}}$ be reduced to quantities that shall have the common index $\frac{1}{8}$. Ans. $16^{\frac{1}{8}}$ and $16^{\frac{1}{8}}$.

4. Let a^2 and $a^{\frac{1}{2}}$ be reduced to quantities that shall have the common index $\frac{1}{4}$. Ans. $(a^8)^{\frac{1}{4}}$ and $(a^2)^{\frac{1}{4}}$.

5. Let $a^{\frac{1}{2}}$ and b be reduced to quantities that shall have the common index $\frac{1}{8}$. Ans. $(a^4)^{\frac{1}{8}}$ and $(b^8)^{\frac{1}{8}}$.

6. Reduce $3^{\frac{1}{2}}$ and $4^{\frac{1}{3}}$ to quantities having a common index.

Ans. $27^{\frac{1}{6}}$ and $16^{\frac{1}{6}}$.

7. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$ to quantities that shall have a common index.

Ans. $(4^4)^{\frac{1}{12}}$ and $(5^3)^{\frac{1}{12}}$.

8. Reduce $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$ to quantities that shall have a common index.

Ans. $(a^3)^{\frac{1}{6}}$ and $(a^2)^{\frac{1}{6}}$.

9. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{4}}$ to quantities that shall have a common index.

Ans. $(a^4)^{\frac{1}{12}}$ and $(b^3)^{\frac{1}{12}}$.

10. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to quantities that shall have a common index.

Ans. $(a^m)^{\frac{1}{mn}}$ and $(b^n)^{\frac{1}{mn}}$.

CASE III.

To reduce an irrational quantity or surd to its most simple form.

Since $a\sqrt[n]{b^nc^nd} = a \times \sqrt[n]{b^n} \times \sqrt[n]{c^n} \times \sqrt[n]{d} = a \times b \times c \times \sqrt[n]{d} = abc\sqrt[n]{d}$,

and $a\sqrt[n]{\frac{b^n}{c}} = a\sqrt[n]{\left(\frac{b^n \times c^{n-1}}{c \times c^{n-1}}\right)} = a\sqrt[n]{\left(\frac{b^n}{c^n} \times c^{n-1}\right)} = a \times \frac{b}{c}\sqrt[n]{c^{n-1}}$

$= \frac{ab}{c}\sqrt[n]{c^{n-1}}$, we have the following

Rule.—Find all the factors in the irrational part of which the required roots can be extracted; and take the remaining factor or product of the remaining factors for the simplified surd; then the product of the coefficient of the given surd and the several roots will be the coefficient of the surd in its simplest form.

When the surd is fractional, multiply both numerator and denominator by such a quantity or number as will render the denominator a power whose root can be extracted; then the coefficient of the surd divided by the root of the denominator will be the coefficient of the simplified surd. If the surd part admits of further reduction, proceed as in the former part of the rule.

EXAMPLES.

1. Reduce $2\sqrt{48}$ and $3\sqrt[3]{108}$ to their most simple forms.

$$2\sqrt{48} = 2\sqrt{16 \times 3} = 2\sqrt{16} \times \sqrt{3} = 2 \times 4 \times \sqrt{3} = 8\sqrt{3}.$$
$$3\sqrt[3]{108} = 3\sqrt[3]{27 \times 4} = 3\sqrt[3]{27} \times \sqrt[3]{4} = 3 \times 3 \times \sqrt[3]{4} = 9\sqrt[3]{4}$$

2. Reduce $\sqrt{\frac{2}{7}}$ and $3\sqrt[3]{\frac{2}{5}}$ to their most simple forms.

$$\sqrt{\tfrac{2}{7}} = \sqrt{\frac{2 \times 7}{7^2}} = \sqrt{\frac{1}{7^2} \times 14} = \sqrt{\frac{1}{7^2}} \times \sqrt{14} = \frac{1}{7}\sqrt{14}.$$
$$3\sqrt[3]{\tfrac{2}{5}} = 3\sqrt[3]{\frac{2 \times 5^2}{5^3}} = 3\sqrt[3]{\frac{1}{5^3} \times 50} = 3 \times \tfrac{1}{5} \times \sqrt[3]{50} = \tfrac{3}{5}\sqrt[3]{50}.$$

3. Reduce $\sqrt[3]{472392}$ and $\sqrt[4]{41472}$ to their most simple forms.

Ans. $54\sqrt[3]{3}$, and $12\sqrt[4]{2}$.

4. Reduce $\sqrt{125}$ and $\sqrt{294}$ to their simplest forms.

$5\sqrt{5}$, and $7\sqrt{6}$.

5. Reduce $\sqrt[3]{56}$ and $\sqrt[3]{192}$ to their most simple forms.

Ans. $2\sqrt[3]{7}$, and $4\sqrt[3]{3}$.

6. Reduce $7\sqrt{80}$ and $9\sqrt[3]{81}$ to their most simple forms.

Ans. $28\sqrt{5}$, and $27\sqrt[3]{3}$.

7. Reduce $\frac{3}{121}\sqrt{\frac{5}{6}}$ and $\frac{4}{7}\sqrt[3]{\frac{3}{16}}$ to their most simple forms.

Ans. $\frac{1}{242}\sqrt{30}$, and $\frac{1}{7}\sqrt[3]{12}$.

8. Reduce $\sqrt{(98a^2x)}$ and $\sqrt{(x^3 - ax^2)}$ to their simplest forms.

Ans. $7a\sqrt{2x}$, and $x\sqrt{x - a}$.

9. Reduce $5\sqrt[3]{4\frac{1}{20}}$ and $\frac{5}{2}\sqrt[4]{9\frac{3}{5}}$ to their most simple forms.

Ans. $\frac{3}{2}\sqrt[3]{150}$, and $\sqrt[4]{375}$.

10. Reduce $\sqrt{3a^2 - 6ax + 3x^2}$ and $\sqrt{(a + x)(a^2 - x^2)}$ to their most simple forms. Ans. $(a - x)\sqrt{3}$, and $(a + x)\sqrt{a - x}$.

11. Reduce $\frac{a}{b}\left(\frac{x^{2n}b^{3n}}{a^n y^{2n}}\right)^{\frac{1}{n}}$, and $\left\{\frac{(a + b)^m (x - y)^{2m}}{(c + d)^{4m}}\right\}^{\frac{1}{2m}}$ to their simplest forms.

Ans. $\frac{b^2x^2}{y^2}$, and $\frac{x - y}{(c + d)^2}(a + b)^{\frac{1}{2}}$.

CASE IV.

To add surd quantities together.

Rule.—When the surds are of the same kind, reduce them to their simplest forms, as in the last case; then if the surd part be the same in them all, annex it to the sum of the rational parts, and it will give the whole sum required.

But if the quantities have different indices, or the surd part be not the same in each of them, they can only be added together by the signs + and −.

EXAMPLES.

1. It is required to find the sum of $\sqrt{27}$ and $\sqrt{48}$.

$$\begin{aligned}\text{Here } \sqrt{27} &= \sqrt{(9 \times 3)} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}\\ \sqrt{48} &= \sqrt{(16 \times 3)} = \sqrt{16} \times \sqrt{3} = \underline{4\sqrt{3}}\\ &\therefore \text{sum} = 7\sqrt{3}.\end{aligned}$$

2. It is required to find the sum of $\sqrt[3]{500}$ and $\sqrt[3]{108}$.

$$\begin{aligned}\text{Here } \sqrt[3]{500} &= \sqrt[3]{(125 \times 4)} = \sqrt[3]{125} \times \sqrt[3]{4} = 5\sqrt[3]{4}\\ \sqrt[3]{108} &= \sqrt[3]{(27 \times 4)} = \sqrt[3]{27} \times \sqrt[3]{4} = \underline{3\sqrt[3]{4}}\\ &\therefore \text{sum} = 8\sqrt[3]{4}.\end{aligned}$$

3. It is required to find the sum of $4\sqrt{147}$ and $3\sqrt{75}$.

$$\begin{aligned}\text{Here } 4\sqrt{147} &= 4\sqrt{(49 \times 3)} = 4 \times \sqrt{49} \times \sqrt{3} = 28\sqrt{3}\\ 3\sqrt{75} &= 3\sqrt{(25 \times 3)} = 3 \times \sqrt{25} \times \sqrt{3} = \underline{15\sqrt{3}}\\ &\therefore \text{sum} = 43\sqrt{3}.\end{aligned}$$

4. It is required to find the sum of $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$.

$$\begin{aligned}\text{Here } 3\sqrt{\frac{2}{5}} &= 3\sqrt{\frac{2 \times 5}{5^2}} = 3\sqrt{\left(\frac{1}{5^2} \times 10\right)} = \frac{3}{5}\sqrt{10}\\ 2\sqrt{\frac{1}{10}} &= 2\sqrt{\frac{1 \times 10}{10^2}} = 2\sqrt{\left(\frac{1}{10^2} \times 10\right)} = \underline{\frac{1}{5}\sqrt{10}}\\ &\therefore \text{sum} = \frac{4}{5}\sqrt{10}.\end{aligned}$$

5. Find the sum of $\sqrt{72}$ and $\sqrt{128}$. Ans. $14\sqrt{2}$.

6. Find the sum of $\sqrt{180}$ and $\sqrt{405}$.	Ans. $15\sqrt{5}$.
7. Find the sum of $3\sqrt[3]{40}$ and $\sqrt[3]{135}$.	Ans. $9\sqrt[3]{5}$.
8. Find the sum of $4\sqrt[3]{54}$ and $5\sqrt[3]{128}$.	Ans. $32\sqrt[3]{2}$.
9. Find the sum of $9\sqrt{243}$ and $10\sqrt{363}$.	Ans. $191\sqrt{3}$.
10. Find the sum of $3\sqrt{\frac{2}{3}}$ and $7\sqrt{\frac{27}{50}}$.	Ans. $\frac{31}{10}\sqrt{6}$.
11. Find the sum of $12\sqrt[3]{\frac{1}{4}}$ and $3\sqrt[3]{\frac{1}{32}}$.	Ans. $6\frac{3}{4}\sqrt[3]{2}$.
12. Find the sum of $\frac{1}{2}\sqrt{(a^2b)}$ and $\frac{1}{3}\sqrt{(4bx^4)}$.	Ans. $\left(\frac{1}{2}a + \frac{2}{3}x^2\right)\sqrt{b}$.

CASE V.

To find the difference of surd quantities.

Rule.—When the surds are of the same kind, prepare the quantities as in the last rule; then the difference of the rational parts annexed to the common surd will give the whole difference required.

But if the quantities have different indices, or the surd part be not the same in each of them, they can only be subtracted by means of the sign $-$.

EXAMPLES.

1. It is required to find the difference of $\sqrt{448}$ and $\sqrt{112}$.

$$\begin{aligned} \text{Here } \sqrt{448} &= \sqrt{(64 \times 7)} = 8\sqrt{7} \\ \text{And } \sqrt{112} &= \sqrt{(16 \times 7)} = 4\sqrt{7} \\ \therefore \text{difference} &= 4\sqrt{7}. \end{aligned}$$

2. It is required to find the difference of $\sqrt[3]{192}$ and $\sqrt[3]{24}$.

$$\begin{aligned} \text{Here } \sqrt[3]{192} &= \sqrt[3]{(64 \times 3)} = 4\sqrt[3]{3} \\ \text{And } \sqrt[3]{24} &= \sqrt[3]{(8 \times 3)} = 2\sqrt[3]{3} \\ \therefore \text{difference} &= 2\sqrt[3]{3}. \end{aligned}$$

3. It is required to find the difference of $5\sqrt{20}$ and $3\sqrt{45}$.

$$\begin{aligned} \text{Here } 5\sqrt{20} &= 5\sqrt{(4 \times 5)} = 10\sqrt{5} \\ \text{And } 3\sqrt{45} &= 3\sqrt{(9 \times 5)} = 9\sqrt{5} \\ \therefore \text{difference} &= \sqrt{5}. \end{aligned}$$

4. Find the difference of $\frac{3}{4}\sqrt{\frac{2}{3}}$ and $\frac{2}{5}\sqrt{\frac{1}{6}}$.

$$\text{Here } \tfrac{3}{4}\sqrt{\tfrac{2}{3}} = \tfrac{3}{4}\sqrt{\tfrac{6}{9}} = \tfrac{3}{12}\sqrt{6} = \tfrac{1}{4}\sqrt{6}$$
$$\text{And } \tfrac{2}{5}\sqrt{\tfrac{1}{6}} = \tfrac{2}{5}\sqrt{\tfrac{6}{36}} = \tfrac{2}{30}\sqrt{6} = \tfrac{1}{15}\sqrt{6}$$
$$\therefore \text{ difference} = \tfrac{11}{60}\sqrt{6}.$$

5. Find the difference of $2\sqrt{50}$ and $\sqrt{18}$. Ans. $7\sqrt{2}$.

6. Find the difference of $\sqrt[3]{320}$ and $\sqrt[3]{40}$. Ans. $2\sqrt[3]{5}$.

7. Find the difference of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$. Ans. $\frac{4}{45}\sqrt{15}$.

8. Find the difference of $2\sqrt{\frac{1}{2}}$ and $\sqrt{8}$. Ans. $\sqrt{2}$.

9. Find the difference of $3\sqrt[3]{\frac{1}{3}}$ and $\sqrt[3]{72}$. Ans. $\sqrt[3]{9}$.

10. Find the difference of $\sqrt[3]{\frac{2}{3}}$ and $\sqrt[3]{\frac{9}{32}}$. Ans. $\frac{1}{12}\sqrt[3]{18}$.

11. Find the difference of $\sqrt{(80a^4x)}$ and $\sqrt{(20a^2x^3)}$.
Ans. $(4a^2-2ax)\sqrt{5x}$.

12. Find the difference of $8\sqrt[3]{(a^3b)}$ and $2\sqrt[3]{(a^6b)}$. Ans. $(8a-2a^2)\sqrt[3]{b}$.

CASE VI.

To multiply surd quantities together.

Rule.—When the surds are of the same kind, multiply the product of the rational parts by the product of the surds, and the result will give the required product, which may be simplified, if necessary, by Case III.

But if the surds are of different kinds, they must be reduced to a common index, and then multiplied together in the usual manner.

EXAMPLES.

1. It is required to find the product of $3\sqrt{8}$ and $2\sqrt{6}$.

$$\text{Here } 3\sqrt{8} \times 2\sqrt{6} = 3 \times 2 \times \sqrt{8} \times \sqrt{6} = 6\sqrt{48} = 6\sqrt{(16 \times 3)}$$
$$= 24\sqrt{3}.$$

2. It is required to find the product of $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$.

$$\text{Here } \tfrac{1}{2}\sqrt[3]{\tfrac{2}{3}} \times \tfrac{3}{4}\sqrt[3]{\tfrac{5}{6}} = \tfrac{1}{2} \times \tfrac{3}{4} \times \sqrt[3]{(\tfrac{2}{3} \times \tfrac{5}{6})} = \tfrac{3}{8}\sqrt[3]{\tfrac{5}{9}} = \tfrac{3}{8}\sqrt[3]{\left(\frac{5 \times 3}{27}\right)}$$
$$= \tfrac{3}{8}\sqrt[3]{(\tfrac{1}{27} \times 15)} = \tfrac{3}{8} \times \tfrac{1}{3}\sqrt[3]{15} = \tfrac{1}{8}\sqrt[3]{15}.$$

3. It is required to find the product of $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$.

$$\text{Here } 2^{\frac{1}{2}} \times 3^{\frac{1}{3}} = 2^{\frac{3}{6}} \times 3^{\frac{2}{6}} = (2^3)^{\frac{1}{6}} \times (3^2)^{\frac{1}{6}} = (8 \times 9)^{\frac{1}{6}} = 72^{\frac{1}{6}}.$$

4. It is required to find the product of $5\sqrt{a}$ and $3\sqrt[3]{a}$.

$$\text{Here } 5\sqrt{a} \times 3\sqrt[3]{a} = 5 \times 3 \times a^{\frac{1}{2}} \times a^{\frac{1}{3}} = 15 \times a^{\frac{3}{6}} \times a^{\frac{2}{6}}$$

$$= 15a^{\frac{3}{6}+\frac{2}{6}} = 15a^{\frac{5}{6}}.$$

5. Find the product of $5ab^2\sqrt{2a}-4a^2b\sqrt{b}$ and $3b\sqrt{2b}+2b\sqrt{a}$.

$$\begin{array}{l} 5ab^2\sqrt{2a}-4a^2b\sqrt{b} \\ 3b\sqrt{2b}\;+2b\sqrt{a} \\ \hline 30ab^3\sqrt{ab}-12a^2b^3\sqrt{2} \\ \qquad\qquad\qquad +10a^2b^3\sqrt{2}-8a^2b^2\sqrt{ab} \\ \hline (30ab^3-8a^2b^2)\sqrt{ab}-2a^2b^3\sqrt{2} \\ \text{or, } 2ab^2(15b-4a)\sqrt{ab}-2a^2b^3\sqrt{2} = \text{product.} \end{array}$$

6. Find the product of $5\sqrt[3]{8}$ and $3\sqrt[3]{5}$. Ans. $30\sqrt[3]{5}$.

7. Find the product of $\sqrt[3]{18}$ and $5\sqrt[3]{4}$. Ans. $10\sqrt[3]{9}$.

8. Required the product of $\frac{1}{4}\sqrt[3]{6}$ and $\frac{2}{15}\sqrt[3]{9}$. Ans. $\frac{1}{10}\sqrt[3]{2}$.

9. Find the product of $\frac{1}{2}\sqrt[3]{18}$ and $5\sqrt[3]{20}$. Ans. $5\sqrt[3]{45}$.

10. Required the product of $2\sqrt{3}$ and $13\frac{1}{2}\sqrt[3]{5}$. Ans. $27\sqrt[6]{675}$.

11. Required the product of $72\frac{1}{4}\sqrt[3]{a^2}$ and $120\frac{1}{2}\sqrt[4]{a}$.

Ans. $8706\frac{1}{8}a^{\frac{11}{12}}$.

12. Required the product of $4 + 2\sqrt{2}$ and $2 - \sqrt{2}$. Ans. 4.

13. Required the product of $(a + b)^{\frac{1}{n}}$ and $(a+ b)^{\frac{1}{m}}$.

Ans. $(a + b)^{\frac{m+n}{mn}}$.

CASE VII.

To divide one surd quantity by another.

Rule.—When the surds are of the same kind, find the quotient of the rational parts, and the quotient of the surds, and the product of these will give the whole quotient required.

But if the surds are of different kinds, they must be reduced to a common index, and then be divided as before.

EXAMPLES.

1. It is required to divide $8\sqrt{108}$ by $2\sqrt{6}$.

Here $\dfrac{8\sqrt{108}}{2\sqrt{6}} = 4\sqrt{18} = 4\sqrt{(9 \times 2)} = 12\sqrt{2}$.

2. It is required to divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

Here $\dfrac{8\sqrt[3]{512}}{4\sqrt[3]{2}} = 2\sqrt[3]{256} = 2\sqrt[3]{(64 \times 4)} = 8\sqrt[3]{4}$.

3. It is required to divide $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt{2}$.

Here $\dfrac{\frac{1}{2}\sqrt{5}}{\frac{1}{3}\sqrt{2}} = \dfrac{3}{2}\sqrt{\dfrac{5}{2}} = \dfrac{3}{2}\sqrt{\dfrac{10}{4}} = \dfrac{3}{4}\sqrt{10}$.

4. It is required to divide $\sqrt{7}$ by $\sqrt[3]{7}$.

Here $\dfrac{\sqrt{7}}{\sqrt[3]{7}} = \dfrac{7^{\frac{1}{2}}}{7^{\frac{1}{3}}} = \dfrac{7^{\frac{3}{6}}}{7^{\frac{2}{6}}} = 7^{\frac{3}{6} - \frac{2}{6}} = 7^{\frac{1}{6}}$.

5. It is required to divide $6\sqrt{54}$ by $3\sqrt{2}$. Ans. $6\sqrt{3}$.

6. It is required to divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$. Ans. $2\sqrt[3]{4}$.

7. Divide $5\frac{3}{4}\sqrt{\frac{1}{135}}$ by $\frac{2}{3}\sqrt{\frac{1}{5}}$. Ans. $\frac{23}{24}\sqrt{3}$.

8. Divide $3\frac{5}{7}\sqrt[3]{\frac{2}{3}}$ by $2\frac{2}{5}\sqrt[3]{\frac{3}{4}}$. Ans. $\frac{65}{63}\sqrt[3]{3}$.

9. Divide $4\frac{1}{2}\sqrt{a}$ by $2\frac{2}{3}\sqrt[3]{(ab)}$. Ans. $\dfrac{27}{16}\left(\dfrac{a}{b^2}\right)^{\frac{1}{6}}$.

10. Divide $32\frac{2}{5}\sqrt{a}$ by $13\frac{3}{4}\sqrt[3]{a}$. Ans. $\frac{648}{275}a^{\frac{1}{6}}$.

11. Divide $9\frac{3}{8}a^{\frac{1}{n}}$ by $4\frac{2}{11}a^{\frac{1}{m}}$. Ans. $\frac{825}{368}a^{\frac{m-n}{mn}}$.

12. Divide $\sqrt{20} + \sqrt{12}$ by $\sqrt{5} - \sqrt{3}$. Ans. $8 + 2\sqrt{15}$.

REMARKS.

1. Since the division of surds is performed by subtracting the index of the divisor from that of the dividend, any power in the numerator of a fractional quantity may be removed into the denominator, and *vice versa*, by simply changing the sign of its index.

Thus, since $a^m \div a^m = a^{m-m} = a^0$, and $a^m \div a^m = 1, \therefore a^0 = 1$;

$$\frac{1}{a} = \frac{1 \times a^{-1}}{a \times a^{-1}} = \frac{a^{-1}}{a^0} = \frac{a^{-1}}{1} = a^{-1}.$$

$$\text{and } \frac{1}{a^n} = \frac{1 \times a^{-n}}{a^n \times a^{-n}} = \frac{a^{-n}}{a^0} = \frac{a^{-n}}{1} = a^{-n}.$$

Similarly $\frac{b}{a^2} = ba^{-2}$; $\frac{a^{-n}}{b^{-n}} = \frac{b^n}{a^n}$; $\frac{1}{a^2} = a^{-2}$; $a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}}$;

$\frac{1}{a+x} = (a+x)^{-1}$; and $a(a^2 - x^2)^{-\frac{1}{3}} = \frac{a}{(a^2 - x^2)^{\frac{1}{3}}}$; and so on.

2. Since $0 \times a$ or $a \times 0 = 0$; therefore $\frac{0}{a} = 0$, and $\frac{0}{0} = a$, where a is any finite quantity whatever.

3. Since $\frac{a}{\frac{1}{2}} = 2a$, $\frac{a}{\frac{1}{3}} = 3a$, $\frac{a}{\frac{1}{4}} = 4a$, *etc.*, it is obvious that the less the denominator is, the greater will be the quotient; therefore if the denominator be indefinitely small, the quotient will be indefinitely great; hence

$$\frac{a}{0} \text{ or } \frac{x}{0}, \text{ or } \frac{1}{0} = \infty.$$

CASE VIII.

To involve or raise surd quantities to any power.

Rule.—When the surd is a simple quantity, multiply its index by 2 for the square, by 3 for the cube, &c., and it will give the power of the surd part, which, being annexed to the proper power of the rational part, will give the whole power required. And if it be a compound quantity, multiply it by itself the proper number of times, according to the usual rule.*

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* When any quantity that is affected with the sign of the square root is to

EXAMPLES.

1. Find the square of $\frac{2}{3}a^{\frac{1}{3}}$, and the cube of $\frac{2}{3}\sqrt{3}$.

$$\text{Here } \left(\frac{2}{3}a^{\frac{1}{3}}\right)^2=\frac{4}{9}a^{\frac{1}{3}\times 2}=\frac{4}{9}a^{\frac{2}{3}}=\frac{4}{9}\sqrt[3]{a^2}.$$

$$\left(\frac{2}{3}\sqrt{3}\right)^3=\frac{8}{27}\times 3^{\frac{3}{2}}=\frac{8}{27}\sqrt{27}=\frac{8}{27}\sqrt{(9\times 3)}=\frac{8}{9}\sqrt{3}.$$

2. Find the square of $3\sqrt[3]{3}$, and the cube of $17\sqrt{21}$.

Ans. $9\sqrt[3]{3^2}$, and $103173\sqrt{21}$.

3. Find the 4th power of $\frac{1}{6}\sqrt{6}$, and the square of $3 + 2\sqrt{5}$.

Ans. $\frac{1}{36}$, and $29 + 12\sqrt{5}$.

4. Find the cube of $\sqrt{x} + 3\sqrt{y}$, and the 4th power of $\sqrt{3} - \sqrt{2}$.

Ans. $x\sqrt{x} + 9x\sqrt{y} + 27y\sqrt{x} + 27y\sqrt{y}$, and $49 - 20\sqrt{6}$.

CASE IX.

To find the roots of surd quantities.

Rule.—When the surd is a simple quantity, multiply its index by $\frac{1}{2}$ for the square root, by $\frac{1}{3}$ for the cube root, &c., and it will give the root of the surd part; which being annexed to the root of the rational part, will give the whole root required. And if it be a compound quantity, find its root by the usual rule.*

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be raised to the second power, or squared, it is done by suppressing the sign. Thus, $(\sqrt{a})^2$, or $\sqrt{a}\times\sqrt{a}=a$; and $\left(\sqrt{(a+b)}\right)^2$, or $\sqrt{(a+b)}\times\sqrt{(a+b)}=a+b$.

* Since the nth root of the mth power of any number a, or the mth power of the nth root of a, is $a^{\frac{m}{n}}$; and the nth root of the mth root of any number a, or the mth root of the nth root of a, is $a^{\frac{1}{mn}}$, it appears that the square root of the square root of a is the 4th root of a; and that the cube root of the square root of a, or the square root of the cube root of a, is the 6th root of a; and so on for the fourth, fifth, or any other numerical root of this kind.

EXAMPLES.

1. It is required to find the square root of $9\sqrt[3]{3}$, and the cube root of $\frac{1}{8}\sqrt{2}$.

$$\text{Here } (9\sqrt[3]{3})^{\frac{1}{2}}=9^{\frac{1}{2}}\times 3^{\frac{1}{3}\times\frac{1}{2}}=9^{\frac{1}{2}}\times 3^{\frac{1}{6}}=3\sqrt[6]{3},$$

$$\text{and } \left(\frac{1}{8}\sqrt{2}\right)^{\frac{1}{3}}=\left(\frac{1}{8}\right)^{\frac{1}{3}}\times(2^{\frac{1}{2}\times\frac{1}{3}})=\frac{1}{2}(2^{\frac{1}{6}})=\frac{1}{2}\sqrt[6]{2}.$$

2. Find the square root of 10^3, and the cube root of $\frac{8}{27}a^4$

Ans. $10\sqrt{10}$, and $\frac{2}{3}a\sqrt[3]{a}$.

3. Find the 4th root of $\frac{16}{81}a^{\frac{2}{3}}$, and the cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$.

Ans. $\frac{2}{3}a^{\frac{1}{6}}$, and $\frac{1}{3}\sqrt{3a}$.

4. Find the square roots of $x^2 - 4x\sqrt{a} + 4a$, and $a + 2\sqrt{ab} + b$.

Ans. $x - 2\sqrt{a}$, and $\sqrt{a} + \sqrt{b}$.

CASE X.

To transform a binomial or residual surd into an entire surd form.

Rule.—Involve the given binomial or residual surd to the second, third, or fourth power, according as the entire surd is to be expressed in the form of the square root, cube root, or fourth root.

EXAMPLES.

1. Transform $2 + \sqrt{3}$ and $\sqrt{2} + \sqrt{5}$ to entire surds of the form of the square root.

$$\text{Here } 2+\sqrt{3}=\sqrt{(2+\sqrt{3})^2}=\sqrt{(4+3+4\sqrt{3})}=\sqrt{(7+4\sqrt{3})},$$
$$\text{and } \sqrt{2}+\sqrt{5}=\sqrt{(\sqrt{2}+\sqrt{5})^2}=\sqrt{(2+5+2\sqrt{10})}=\sqrt{(7+2\sqrt{10})}.$$

The binomial surd $2+\sqrt{3}$ may also be expressed as an entire surd of the form of the *cube root.* Thus,

$$2+\sqrt{3}=\sqrt[3]{(2+\sqrt{3})^3}=\sqrt[3]{(8+12\sqrt{3}+18+3\sqrt{3})}=\sqrt[3]{(26+15\sqrt{3})}.$$

2. Transform $\sqrt[3]{2} + \sqrt[3]{4}$ to an entire surd.

$$(\sqrt[3]{2}+\sqrt[3]{4})^3 = 2 + 3\sqrt[3]{16} + 3\sqrt[3]{32} + 4 = 6 + 6\sqrt[3]{2} + 6\sqrt[3]{4};$$

$$\text{hence } \sqrt[3]{2}+\sqrt[3]{4} = \{6(1+\sqrt[3]{2}+\sqrt[3]{4})\}^{\frac{1}{3}}.$$

3. Transform $3 - \sqrt{5}$ and $\sqrt{2} - 2\sqrt{6}$ to entire surds.

Ans. $\sqrt{(14 - 6\sqrt{5})}$, and $\sqrt{(26 - 8\sqrt{3})}$.

4. Transform $4 - \sqrt{7}$ and $2\sqrt[3]{3} - \sqrt[3]{9}$ to entire surds.

Ans. $\sqrt{(23 - 8\sqrt{7})}$ and $\{15 - 18(2\sqrt[3]{3} - \sqrt[3]{9})\}^{\frac{1}{3}}$.

CASE XI.

To extract the square root of a binomial surd, one of whose terms is rational, and the other a quadratic surd.

Since by involution $(\sqrt{x} \pm \sqrt{y})^2 = x + y \pm 2\sqrt{xy}$; therefore inversely

$$\sqrt{(x + y \pm 2\sqrt{xy})} = \sqrt{x} \pm \sqrt{y}.$$

Now when x and y are numerical quantities, they can be so incorporated that their sum shall be one numerical quantity, and their product another, and then all trace of their values is lost. We have thus an expression of the form $a \pm \sqrt{b}$, where $a = x + y$, and $\sqrt{b} = 2\sqrt{xy}$ or $\sqrt{4xy}$.

The values of x and y may, however, be discovered by the following process:—*

Let $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$; then, by squaring, we have

$$a + \sqrt{b} = x + y + 2\sqrt{xy}; \quad \therefore a = x + y, \text{ and } \sqrt{b} = \sqrt{4xy}.$$

Squaring these last two expressions of equality, we have

$$x^2 + 2xy + y^2 = a^2$$

$$4xy \quad = b,$$

$\therefore$ by subtraction $x^2 - 2xy + y^2 = a^2 - b = c^2$, suppose,

and taking the root, $x - y = \sqrt{a^2 - b}$ or c.

But, $x + y = a$;

hence adding and dividing by 2, gives $x = \dfrac{a + \sqrt{a^2 - b}}{2} = \dfrac{a + c}{2}$;

and subtracting, and dividing by 2, gives $y = \dfrac{a - \sqrt{a^2 - b}}{2} = \dfrac{a - c}{2}$;

$$\therefore \sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y} = \sqrt{\frac{a + c}{2}} + \sqrt{\frac{a - c}{2}}.$$

Similarly $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y} = \sqrt{\dfrac{a + c}{2}} - \sqrt{\dfrac{a - c}{2}}$.

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* This investigation may be deferred till a knowledge of equations is obtained.

In this manner the square root of a binomial surd may be obtained, and the entire process should be gone through in almost every instance, as little real benefit can be derived by the student from mere substitution in any formula.

EXAMPLES.

1. Extract the square root of $11 + 6\sqrt{2}$.

Let $\sqrt{11 + 6\sqrt{2}} = \sqrt{x} + \sqrt{y}$, $\therefore$ $11 + 6\sqrt{2} = x + y + 2\sqrt{xy}$;
hence $x + y = 11$ and $2\sqrt{xy} = 6\sqrt{2}$, or, by squaring,

$$x^2 + 2xy + y^2 = 121$$
$$4xy = 72,$$

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 49$
and the square root gives, $x - y = 7$,
but $x + y = 11$;
hence $2x = 18$ or $x = 9$ and $2y = 4$ or $y = 2$.

Therefore $\sqrt{11 + 6\sqrt{2}} = \sqrt{x} + \sqrt{y} = \sqrt{9} + \sqrt{2} = 3 + \sqrt{2}$.
Similarly $\sqrt{11 - 6\sqrt{2}} = \sqrt{x} - \sqrt{y} = \sqrt{9} - \sqrt{2} = 3 - \sqrt{2}$.

2. Find the square roots of $3 - 2\sqrt{2}$, $6 \pm 2\sqrt{5}$ and $23 \pm 8\sqrt{7}$.
Ans. $\sqrt{2} - 1$, $\sqrt{5} \pm 1$, and $4 \pm \sqrt{7}$.

3. Find the square roots of $36 \pm 10\sqrt{11}$, $33 \pm 12\sqrt{6}$ and $94 \pm 42\sqrt{5}$.
Ans. $5 \pm \sqrt{11}$, $2\sqrt{6} \pm 3$, and $7 \pm 3\sqrt{5}$.

4. Find the square roots of $\frac{7 + 3\sqrt{5}}{2}$, $\frac{47 - 21\sqrt{5}}{10}$, and $\frac{14 \pm 5\sqrt{3}}{2}$.
Ans. $\frac{3 + \sqrt{5}}{2}$—$\frac{15 - 7\sqrt{5}}{10}$, and $\frac{5 \pm \sqrt{3}}{2}$.

CASE XII.

To find a multiplier, or multipliers, so that the product of a binomial or trinomial surd by such multiplier or multipliers, may become rational.

We have already seen, pages 7 and 16, that

$$(x + y)(x - y) = x^2 - y^2,$$
$$(x + y)(x^2 - xy + y^2) = x^3 + y^3,$$
$$(x - y)(x^2 + xy + y^2) = x^3 - y^3;$$

and in a similar manner we find that

$$(x + \sqrt{y})(x - \sqrt{y}) = x^2 - y \quad . \quad . \quad . \quad (1)$$
$$(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) = x - y, \quad . \quad . \quad . \quad (2)$$
$$(\sqrt[3]{x} + \sqrt[3]{y})(\sqrt[3]{x^2} - \sqrt[3]{xy} + \sqrt[3]{y^2}) = x + y, \quad . \quad . \quad . \quad (3)$$
$$(\sqrt[3]{x} - \sqrt[3]{y})(\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}) = x - y. \quad . \quad . \quad . \quad (4).$$

Also, since $(\sqrt[4]{x} + \sqrt[4]{y})(\sqrt[4]{x} - \sqrt[4]{y}) = \sqrt{x} - \sqrt{y}$,
and $(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) = x - y$ . . by (1):
$\therefore (\sqrt[4]{x} + \sqrt[4]{y})(\sqrt[4]{x} - \sqrt[4]{y})(\sqrt{x} + \sqrt{y}) = x - y$ . . . (5).

By means of these and similar formulæ, the requisite multiplier or multipliers may readily be found in all cases that usually occur in practice.

EXAMPLES.

1. Find a multiplier that shall render $5 + \sqrt{3}$ rational.
By formula (1) the multiplier is $5 - \sqrt{3}$, and
$(5 + \sqrt{3})(5 - \sqrt{3}) = 25 - 3 = 22$, a rational product.

2. Find a multiplier that shall render $\sqrt{5} + \sqrt{3}$ rational.
By formula (2) the multiplier is $\sqrt{5} - \sqrt{3}$, hence
$(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) = 5 - 3 = 2$, a rational product.

3. Find multipliers that shall render $\sqrt[4]{5} + \sqrt[4]{3}$ rational.
By formula (5) the multipliers are $\sqrt[4]{5} - \sqrt[4]{3}$ and $\sqrt{5}+\sqrt{3}$; hence
$(\sqrt[4]{5}+\sqrt[4]{3})(\sqrt[4]{5}-\sqrt[4]{3}) = \sqrt{5}-\sqrt{3}$, and again
$(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3}) = 5-3 = 2$, a rational product.

4. Find a multiplier that will render $\sqrt[3]{7^2}-\sqrt[3]{7\times 3}+\sqrt[3]{3^2}$ rational.
By formula (3) the multiplier is $\sqrt[3]{7}+\sqrt[3]{3}$; hence
$(\sqrt[3]{7^2}-\sqrt[3]{7\times 3}+\sqrt[3]{3^2})(\sqrt[3]{7}+\sqrt[3]{3}) = 7+3 = 10$, a rational product.

5. Find the multipliers that shall rationalize each of the quantities, $\sqrt{5}-\sqrt{x}$, $\sqrt{a}+\sqrt{b}$, $a+\sqrt{b}$, $1-\sqrt[3]{2a}$, $\sqrt[3]{3}-\frac{1}{2}\sqrt[3]{2}$, and $\sqrt[3]{9}+\sqrt[3]{6}+\sqrt[3]{4}$.
Ans. $\sqrt{5}+\sqrt{x}$, $\sqrt{a}-\sqrt{b}$, $a-\sqrt{b}$, $1+\sqrt[3]{2a}+\sqrt[3]{4a^2}$,
$\sqrt[3]{3^2}+\frac{1}{2}\sqrt[3]{6}+\frac{1}{4}\sqrt[3]{4}$ and $\sqrt[3]{3}-\sqrt[3]{2}$.

CASE XIII.

To reduce a fraction, whose denominator is either a simple or a compound surd, to another that shall have a rational denominator.

Rule.—1. When any simple fraction is of the form $\frac{b}{\sqrt{a}}$, multiply each of its terms by $\sqrt{a}$, and the resulting fraction will be $\frac{b\sqrt{a}}{a}$.

Or when it is of the form $\frac{b}{\sqrt[3]{a}}$, multiply them by $\sqrt[3]{a^2}$, and the result will be $\frac{b\sqrt[3]{a^2}}{a}$.

And for the general form, $\frac{b}{\sqrt[n]{a}}$, multiply by $\sqrt[n]{a^{n-1}}$, and the result will be $\frac{b\sqrt[n]{a^{n-1}}}{a}$.

2. If it be a compound surd, find such a multiplier, by the rule of last case, as will make the denominator rational; and multiply both the numerator and denominator by it, and the result will be the fraction required.

EXAMPLES.

1. Reduce the fractions $\frac{2}{\sqrt{3}}$ and $\frac{3}{\frac{1}{2}\sqrt[4]{5}}$ to others that shall have rational denominators.

$$\text{Here } \frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}\text{; and } \frac{3}{\frac{1}{2}\sqrt[4]{5}}=\frac{3}{\frac{1}{2}\sqrt[4]{5}}\times\frac{\sqrt[4]{5^3}}{\sqrt[4]{5^3}}=$$
$$\frac{3\sqrt[4]{5^3}}{\frac{1}{2}\times 5}=\frac{6\sqrt[4]{5^3}}{5}=\frac{6}{5}\sqrt[4]{125}.$$

2. Reduce $\frac{3}{\sqrt{5}-\sqrt{2}}$ and $\frac{\sqrt{2}}{3-\sqrt{2}}$ to fractions whose denominators shall be rational.

Here

$$\frac{3}{\sqrt{5}-\sqrt{2}}\times\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\frac{3\sqrt{5}+3\sqrt{2}}{5-2}=\frac{3\sqrt{5}+3\sqrt{2}}{3}=\sqrt{5}+\sqrt{2}$$

and

$$\frac{\sqrt{2}}{3-\sqrt{2}}=\frac{\sqrt{2}\times(3+\sqrt{2})}{(3-\sqrt{2})\times(3+\sqrt{2})}=\frac{3\sqrt{2}+2}{9-2}=\frac{2+3\sqrt{2}}{7}=\frac{2}{7}+\frac{3}{7}\sqrt{2}.$$

3. Reduce $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}$, $\frac{x}{3+\sqrt{x}}$, $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ and $\frac{10}{\sqrt[3]{7}-\sqrt[3]{5}}$ to equivalent fractions having rational denominators.

Ans. $\frac{\sqrt{42}-3\sqrt{2}}{4}$, $\frac{3x-x\sqrt{x}}{9-x}$, $\frac{a+b-2\sqrt{ab}}{a-b}$ and $5(\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25})$.

4. Reduce $\frac{\sqrt[3]{3}}{\sqrt[3]{9}+\sqrt[3]{10}}$, $\frac{4}{\sqrt[4]{4}+\sqrt[4]{5}}$ and $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$ to equivalent fractions having rational denominators.

Ans. $\frac{3\sqrt[3]{9}-3\sqrt[3]{10}+\sqrt[3]{300}}{19}$, $4(\sqrt[4]{5}-\sqrt[4]{4})(\sqrt{5}+2)$, and $\frac{a+\sqrt{a^2-x^2}}{x}$.

MISCELLANEOUS EXERCISES ON IRRATIONAL QUANTITIES.

	Answers
1. Simplify $\sqrt[3]{40} + \sqrt[3]{135} - \frac{1}{2}\sqrt[3]{320}$.	$3\sqrt[3]{5}$.
2. Simplify $8\sqrt{\frac{3}{4}} - \frac{1}{2}\sqrt{12} + 4\sqrt{27} - 2\sqrt{\frac{3}{16}}$.	$\frac{29}{2}\sqrt{3}$.
3. Simplify $y\sqrt[3]{8x^6y} + 4x\sqrt[3]{x^3y^4} - \sqrt[3]{125x^6y^4}$.	$x^2y^{\frac{4}{3}}$.
4. Simplify $\frac{3}{4}\sqrt[3]{8m^3n} - \frac{2n}{3}\sqrt[3]{\frac{m^6}{27n^5}}$.	$\frac{(27mn - 4m^2)\sqrt[3]{n}}{18n}$.
5. Simplify $\frac{1}{3}\sqrt[3]{a^2} \times 6\sqrt{a^3} \times a^{\frac{1}{6}} \div \sqrt[3]{a^4}$.	$2a$.
6. Simplify $\left(\frac{ax}{y}\right)^{\frac{1}{2}} \times \left(\frac{by}{x^2}\right)^{\frac{1}{3}} \times \left(\frac{x^2}{a^3b^2}\right)^{\frac{1}{6}}$.	$x^{\frac{1}{6}}y^{-\frac{1}{6}}$ or $\left(\frac{x}{y}\right)^{\frac{1}{6}}$.
7. Multiply $a^{\frac{1}{3}} + 2a^{\frac{1}{6}} - 3$ by $a^{\frac{1}{3}} - 2a^{\frac{1}{6}} + 3$.	$a^{\frac{2}{3}} - 4a^{\frac{1}{3}} + 12a^{\frac{1}{6}} - 9$.
8. Multiply $\sqrt{(x^2 + 2y^2)} + y$ by $\sqrt{(x^2 + 2y^2)} - y$.	$x^2 + y^2$.
9. Divide $\sqrt[4]{(x^4 - 3x^3 - 2x^2)}$ by $\sqrt{x}$.	$(x^2 - 3x - 2)^{\frac{1}{4}}$
10. Divide $3a + 6a^{\frac{7}{6}} - a^{\frac{5}{4}} + 2a^{\frac{4}{3}} - 2a^{\frac{17}{12}} + 4a^{\frac{3}{2}}$ by $3a^{\frac{1}{2}} - a^{\frac{3}{4}} + 2a^{\frac{5}{6}}$.	$a^{\frac{1}{2}} + 2a^{\frac{2}{3}}$.
11. Divide $\frac{m^{\frac{1}{2}}n^{\frac{1}{4}} + m^{\frac{1}{4}}n^{\frac{1}{2}}}{m^{\frac{1}{2}}n^{\frac{1}{4}} - m^{\frac{1}{4}}n^{\frac{1}{2}}}$ by $\frac{m^{\frac{1}{4}} + n^{\frac{1}{4}}}{m^{\frac{1}{4}} - n^{\frac{1}{4}}}$.	1.
12. Divide $6x(x - y)\sqrt{y} - xy\sqrt{6x}$ by $2x\sqrt{3} - 3\sqrt{2xy}$.	$x\sqrt{3y} + y\sqrt{2x}$.
13. Simplify $\frac{x^{\frac{5}{2}} + x^{-\frac{1}{2}}}{x^{\frac{5}{2}} + 2x^{\frac{3}{2}} - x^{-\frac{1}{2}}}$;	$\frac{x^2 - x + 1}{x^2 + x - 1}$.
14. Simplify $\frac{xy}{y - m} \pm \left\{\frac{x^2y^2}{(y - m)^2} - \frac{x^2y}{y - m}\right\}^{\frac{1}{2}}$.	$\frac{x(y \pm \sqrt{my})}{y - m}$.
15. Simplify $\left[\{(a^{-1})^{-2}\}^{-3}\right]^{-4}$ and $\{(a^{-\frac{1}{2}})^{-\frac{1}{3}}\}^{-\frac{1}{4}}$.	a^{24} and $a^{-\frac{1}{24}}$.
16. Simplify $(y + \sqrt[3]{x^2y})^2 + (x + \sqrt[3]{xy^2})^2$.	$(x^{\frac{2}{3}} + y^{\frac{2}{3}})^3$.

	Answers.
17. Find the square root of $9m^2n + 4an^2 - 12mn\sqrt{an}$.	$3m\sqrt{n} - 2n\sqrt{a}$.
18. Simplify $\dfrac{m + n\sqrt{-1}}{m - n\sqrt{-1}} + \dfrac{m - n\sqrt{-1}}{m + n\sqrt{-1}}$.	$\dfrac{2(m^2 - n^2)}{m^2 + n^2}$.
19. Simplify $\dfrac{2\sqrt{3} + 2}{\sqrt{3} - \sqrt{2} + 1}$ or $\dfrac{2(\sqrt{3} + 1)}{(\sqrt{3} + 1) - \sqrt{2}}$.	$\sqrt{3} + \sqrt{2} + 1$.
20. If $x = \frac{1}{2}(a + a^{-1})$ and $y = \frac{1}{2}(b + b^{-1})$, what is the value of $xy + (1 - x^2)^{\frac{1}{2}} . (1 - y^2)^{\frac{1}{2}}$.	$\frac{1}{2}\left(ab + \dfrac{1}{ab}\right)$.
21. Simplify $\dfrac{a\sqrt{a + x}}{\sqrt{a + x} + \sqrt{x}} + \dfrac{a\sqrt{a + x}}{\sqrt{a + x} - \sqrt{x}}$.	$2(a + x)$.

IMAGINARY QUANTITIES.

An *imaginary* quantity is one whose value cannot be assigned in consequence of the *impossibility* of performing the operation indicated. Thus $\sqrt{-a^2}$ indicates an operation which cannot be performed, since the square root of $-a^2$ is neither $+a$ nor $-a$, for $(+a)^2 = +a^2$ and $(-a)^2 = +a^2$; and therefore $\sqrt{-a^2}$ is an imaginary or impossible quantity. If, however, we disengage the quantity a^2 from under the radical, we get $\sqrt{-a^2} = \sqrt{a^2 \times (-1)} = \pm a\sqrt{-1}$, and thus it is plain that all imaginary quantities will depend on the single imaginary $\sqrt{-1}$.

$$\begin{aligned}
\text{Now } (\sqrt{-1})^2 &= \{(-1)^{\frac{1}{2}}\} = (-1)^{\frac{2}{2}} = (-1)^1 = -1;\\
(\sqrt{-1})^3 &= (\sqrt{-1})^2 \times \sqrt{-1} = -1 \times \sqrt{-1} = -\sqrt{-1};\\
(\sqrt{-1})^4 &= (\sqrt{-1})^2 \times (\sqrt{-1})^2 = -1 \times -1 = 1\\
(\sqrt{-1})^5 &= (\sqrt{-1})^4 \times (\sqrt{-1}) = 1 \times \sqrt{-1} = \sqrt{-1},
\end{aligned}$$

and so on.

Keeping these results in mind, the various operations to which all real quantities can be subjected, will equally apply to imaginary quantities.

Thus
$$
\begin{aligned}
a\sqrt{-1} + b\sqrt{-1} + c\sqrt{-1} &= (a + b + c)\sqrt{-1},\\
x\sqrt{-1} - 2y\sqrt{-1} + 3z\sqrt{-1} &= (x - 2y + 3z)\sqrt{-1},\\
(x + y\sqrt{-1}) + (a - 2b)\sqrt{-1} &= x + a + (y - 2b)\sqrt{-1}.
\end{aligned}
$$

EXAMPLES.

1. Multiply $x + y\sqrt{-1}$ by $x - y\sqrt{-1}$, and divide $1 + \sqrt{-1}$ by $1 - \sqrt{-1}$.

$$
\begin{array}{l}
x + y\sqrt{-1}\\
\underline{x - y\sqrt{-1}}\\
x^2 + xy\sqrt{-1}\\
\quad - xy\sqrt{-1} - y^2(-1)\\
\hline
\end{array}
$$

$\therefore$ product $= x^2 - y^2(-1) = x^2 + y^2$, a real quantity.

Again $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}} = \dfrac{1+\sqrt{-1}}{1-\sqrt{-1}} \times \dfrac{1+\sqrt{-1}}{1+\sqrt{-1}} = \dfrac{(1+\sqrt{-1})^2}{2} = \dfrac{2\sqrt{-1}}{2}$
$= \sqrt{-1}$.

2. Multiply $2\sqrt{-3}$ by $3\sqrt{-5}$, and divide $-24\sqrt{-15}$ by $6\sqrt{-5}$.

Here $2\sqrt{-3} \times 3\sqrt{-5} = 2\sqrt{3 \times (-1)} \times 3\sqrt{5 \times (-1)}$
$= 6\sqrt{15} \times \sqrt{-1} \times \sqrt{-1} = -6\sqrt{15}$.

And $\dfrac{-24\sqrt{-15}}{6\sqrt{-5}} = -\dfrac{24}{6} \times \dfrac{\sqrt{15 \times (-1)}}{\sqrt{5 \times (-1)}} = -4\sqrt{3} \times \dfrac{\sqrt{-1}}{\sqrt{-1}}$
$= -4\sqrt{3}$.

3. Divide $2 + 3\sqrt{-1}$ by $2 - \sqrt{-1}$, and also $2a + 3b\sqrt{-1}$ by $2a - b\sqrt{-1}$.

Ans. $\dfrac{1 + 8\sqrt{-1}}{5}$ and $\dfrac{4a^2 - 3b^2 + 8ab\sqrt{-1}}{4a^2 + b^2}$.

4. Find the continued product of $x + a\sqrt{-1}$, $x - a\sqrt{-1}$, $x + \frac{a}{2}(3 - \sqrt{-5})$ and $x + \frac{a}{2}(3 + \sqrt{-5})$.

Ans. $x^4 + 3ax^3 + \frac{9}{2}a^2x^2 + 3a^3x + \frac{7}{2}a^4$.

5. Find the fourth power of $1 - 2\sqrt{-1}$, and the cubes of $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$

Ans. $-7 + 24\sqrt{-1}$, 1 and 1.

6. Find the square root of $5 - 12\sqrt{-1}$, and the value of $\sqrt{4 + 6\sqrt{-5}} + \sqrt{4 - 6\sqrt{-5}}$.

Ans. $3 - 2\sqrt{-1}$ and 6.

EQUATIONS.

An *equation* is the expression of the equality of two algebraical quantities, one or both of which contain some power of the unknown quantity. Thus $2x + 3 = 15$, $ax^2 = bx + c$, and $x^3 + ax^2 + bx - c = 0$ are equations.

The *members* or *sides* of an equation are the two quantities between which the sign of equality is placed. Thus in the equation $2x + 3 = 15$, the *first* or *left-hand* member is $2x + 3$, and the *second* or *right-hand* member is 15.

The *terms* of an equation are the several quantities of which it is composed; thus x, a, and b are the terms of the equation $x + a = b$.

A *simple* equation is one which contains only the first power of the unknown quantity, as $2x + 3a = 4x - 3b$.

A *quadratic equation* is one which contains the second power, or both the first and second powers of the unknown quantity, as

$$x^2 = 25,\ x^2 + 10x = 200,\ ax^2 - bx = -c,$$

$$\text{or } x^2 - \frac{b}{a}x = -\frac{c}{a},\ \text{or } x^2 - px = -q.$$

The first of these is a *simple* or *pure quadratic*, and each of the others is a *compound*, or *adfected quadratic.*

A *cubic equation* is one in which the highest power of the unknown quantity is the third power, as $x^3 = 54$, $2x^3 - 3x = 35$, or $x^3 - ax^2 + bx = c$.

A *biquadratic equation* is one in which the highest power of the unknown quantity is the fourth power, as $x^4 = 25$, $5x^4 - 4x = 6$, or $x^4 + ax^3 + bx^2 + cx = d$.

An equation is said to be of the *first*, *second*, *third*, or *n*th degree, according as the highest power of the unknown quantity contained in it, is the *first*, *second*, *third*, or *n*th power of that quantity. Thus,

$$x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \ldots\ldots hx = k,$$

is an equation of the *n*th degree.

A *root of an equation* is that quantity which, when substituted for the unknown quantity in it, makes both members the *same*, and thus verifies the equation.

A simple equation has only one root. For if the equation $2x + 3a = 5b$, can have *two* roots, let them be r_1 and r_2; then by substituting each of these for the unknown quantity x, we have

$$2r_1 + 3a = 5b$$
$$2r_2 + 3a = 5b.$$

Now, if from the first of these equations the second be subtracted, we get

$$2r_1 - 2r_2 = 0, \text{ or } r_1 - r_2 = 0, \therefore r_1 = r_2;$$

and the simple equation has therefore only one root.

An *identical equation* is one which can be verified by all values of the quantities contained in it. Thus $(x-y)^2 = (x+y)^2 - 4xy$ is an identical equation, for if any values whatever be assigned to x and y, the equation will be verified.

The *solution of an equation* is the method of separating the unknown quantity from the other known quantities in the equation, so that the unknown quantity shall form one member of the equation, and a combination of the known quantities the other member.

SIMPLE EQUATIONS

The solution of simple equations, and of those of higher degrees, depends entirely on the following axioms :—

1. If equals, or the same be added to equals, the wholes or sums are equal.
2. If equals, or the same be taken from equals, the remainders are equal.
3. If equals be multiplied by the same, or by equals, the products are equal.
4. If equals be divided by the same, or by equals, the quotients are equal.
5. If equals be involved to the same power, the powers are equal.
6. If of equals the same root be extracted, the roots are equal.

Rule I. Any quantity may be transposed from one side of an equation to the other, by changing its sign from + to −, or from − to +, and the equation will still subsist, as is evident from axioms 1 and 2.

Thus, if $x + 3 = 7$; then will $x = 7 - 3$, or $x = 4$.

And, if $x - 4 + 6 = 8$; then will $x = 8 + 4 - 6 = 6$.

Also, if $x - a + b = c - d$; then will $x = a - b + c - d$.

And, if $4x - 8 = 3x + 20$; then $4x - 3x = 20 + 8$, and consequently $x = 28$.

From this rule it also follows, that if a quantity be found on each side of an equation, with the same sign, it may be left out of both of them; and that the signs of all the terms of any equation may be changed from + to −, or from − to +, without altering its value.

Thus, if $x + 5 = 7 + 5$; then, by cancelling, $x = 7$.

And, if $a - x = b - c$; then, by changing the signs, $x - a = c - b$, or $x = a + c - b$.

Rule II. If, in any equation, the unknown quantity be multiplied by any number or quantity, the multiplier may be taken away, by dividing all the rest of the terms by it; and if the unknown quantity be divided by any number, the divisor may be taken away, by multiplying all the other terms by it, as is evident from axioms 3 and 4.

Thus, if $ax = 3ab - c$; then will $x = 3b - \frac{c}{a}$.

If $2x + 4 = 16$; then will $x + 2 = 8$, or $x = 8 - 2 = 6$.

Also, if $\frac{x}{2} = 5 + 3$; then will $x = 10 + 6 = 16$.

And, if $\frac{2x}{3} - 2 = 4$; then $2x - 6 = 12$, or by division, $x - 3 = 6$ or $x = 9$.

Rule III. Any equation may be cleared of fractions, by multiplying each of its terms, successively, by the denominators of those fractions; or by multiplying both sides by the product of all the denominators, or by the least common multiple of them, as is obvious from axiom 3.

Thus, if $\frac{x}{3} + \frac{x}{4} = 5$, then, multiplying by 3, we have $x + \frac{3x}{4} = 15$; and this, multiplied by 4, gives $4x + 3x = 60$; whence, by addition, $7x = 60$, or $x = \frac{60}{7} = 8\frac{4}{7}$.

And, if $\frac{x}{4} + \frac{x}{6} = 10$; then, multiplying by 12 (which is a multiple of 4 and 6,) $3x + 2x = 120$, or $5x = 120$, or $x = \frac{120}{5} = 24$.

It also appears, from this rule, that if the same number or quantity be found in each of the terms of an equation, either as a multiplier or divisor, it may be expunged from all of them, without altering the result.

Thus, if $ax = ab + ac$; then, by cancelling, $x = b + c$.

And, if $\frac{x}{a} + \frac{b}{a} = \frac{c}{a}$; then, $x + b = c$, or $x = c - b$.

Rule. IV. If the unknown quantity, in any equation, be in the form of a surd, transpose the terms so that this may stand alone, on one side of the equation, and the remaining terms on the other (by *Rule* I.); then involve each of the sides to such a power as corresponds with the index of the surd, and the equation will be rendered free from any irrational expression, as appears from axiom 5.

Thus, if $\sqrt{x} - 2 = 3$; then will $\sqrt{x} = 3 + 2 = 5$, or by squaring, $x = 5^2 = 25$.

And, if $\sqrt{(3x + 4)} = 5$; then will $3x + 4 = 25$, or $3x = 25 - 4 = 21$, and consequently $x = \frac{21}{3} = 7$.

Also, if $\sqrt[3]{(2x + 3)} + 4 = 8$; then will $\sqrt[3]{(2x + 3)} = 8 - 4 = 4$, or $2x + 3 = 4^3 = 64$, and consequently $2x = 64 - 3 = 61$, or $x = \frac{61}{2} = 30\frac{1}{2}$.

Rule. V. If that side of the equation, which contains the unknown quantity, be a complete power, the equation may be reduced to one of a lower degree, by extracting the root of the said power on both sides of the equation, as is obvious from axiom 6.

Thus, if $x^2 = 81$, then $x = \sqrt{81} = 9$; and if $x^3 = 27$, then $x = \sqrt[3]{27} = 3$.

Also, if $3x^2 - 9 = 24$; then $3x^2 = 24 + 9 = 33$, or $x^2 = \frac{33}{3} = 11$, and consequently $x = \sqrt{11}$.

And, if $x^2 + 6x + 9 = 27$; then, since the left-hand side of the equation is a complete square, we shall have, by extracting the roots, $x + 3 = \sqrt{27} = \sqrt{(9 \times 3)} = 3\sqrt{3}$, or $x = 3\sqrt{3} - 3$.

Rule. VI. Any analogy or proportion may be converted into an equation, by making the product of the two extreme terms equal to that of the two means.*

Thus, if $3x : 16 :: 5 : 6$; then $3x \times 6 = 16 \times 5$, or $18x = 80$, or $x = \frac{80}{18} = \frac{40}{9} = 4\frac{4}{9}$.

And, if $\frac{2x}{3} : a :: b : c$; then will $\frac{2cx}{3} = ab$, or $2cx = 3ab$; hence, by division, $x = \frac{3ab}{2c}$.

Also, if $12 - x : \frac{x}{2} :: 4 : 1$; then $12 - x = \frac{4x}{2} = 2x$, or $2x + x = 12$; and consequently $x = \frac{12}{3} = 4$.

MISCELLANEOUS EXAMPLES.

1. Given $5x - 15 = 2x + 6$, to find the value of x.

Here $5x - 2x = 6 + 15$, or $3x = 6 + 15 = 21$; and therefore, by division, we shall have $x = \frac{21}{3} = 7$.

2. Given $40 - 6x - 16 = 120 - 14x$, to find the value of x.

Here $14x - 6x = 120 - 40 + 16$; or $8x = 136 - 40 = 96$; and therefore, by division, $x = \frac{96}{8} = 12$.

3. Given $3x^2 - 10x = 8x + x^2$, to find the value of x.

Here $3x - 10 = 8 + x$, by dividing by x;
or $3x - x = 8 + 10 = 18$, by transposition.

And consequently $2x = 18$ or by division, $x = \frac{18}{2} = 9$.

4. Given $6ax^3 - 12abx^2 = 3ax^3 + 6ax^2$, to find the value of x.

Here $2x - 4b = x + 2$, by dividing by $3ax^2$;
or $2x - x = 2 + 4b$; and therefore $x = 4b + 2$.

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* This rule may be deduced from definitions 9 and 10.

5. Given $x^2 + 2x + 1 = 16$, to find the value of x.

Here $x + 1 = 4$, by extracting the square root of each side; and by transposition, $x = 4 - 1 = 3$.

6. Given $5ax - 3b = 2dx + c$, to find the value of x.

Here $5ax - 2dx = 3b + c$; or $(5a - 2d)x = 3b + c$; and therefore, by division, $x = \dfrac{3b + c}{5a - 2d}$.

7. Given $\dfrac{x}{2} - \dfrac{x}{3} + \dfrac{x}{4} = 10$, to find the value of x.

Here $x - \dfrac{2x}{3} + \dfrac{2x}{4} = 20$; and $3x - 2x + \dfrac{6x}{4} = 60$;
or $12x - 8x + 6x = 240$; whence $10x = 240$, or $x = 24$.

8. Given $\dfrac{x - 3}{2} + \dfrac{x}{3} = 20 - \dfrac{x - 19}{2}$, to find the value of x.

Here $x - 3 + \dfrac{2x}{3} = 40 - x + 19$; or $3x - 9 + 2x = 120 - 3x + 57$; whence $3x + 2x + 3x = 120 + 57 + 9$; that is, $8x = 186$, or $x = 23\frac{1}{4}$.

9. Given $\sqrt{\dfrac{2x}{3}} + 5 = 7$, to find the value of x.

Here $\sqrt{\dfrac{2x}{3}} = 7 - 5 = 2$; whence, by squaring, $\dfrac{2x}{3} = 2^2 = 4$, and $2x = 12$, or $x = 6$.

10. Given $x + \sqrt{(a^2 + x^2)} = \dfrac{2a^2}{\sqrt{(a^2 + x^2)}}$, to find the value of x.

Here $x\sqrt{(a^2 + x^2)} + a^2 + x^2 = 2a^2$, or $x\sqrt{(a^2 + x^2)} = a^2 - x^2$,
$\therefore \quad x^2(a^2 + x^2) = a^4 - 2a^2x^2 + x^4$, or $a^2x^2 + x^4 = a^4 - 2a^2x^2 + x^4$;

$\therefore \quad 3a^2x^2 = a^4$, or $3x^2 = a^2$; hence $x^2 = \dfrac{a^2}{3}$, and therefore

$$x = \sqrt{\frac{a^2}{3}} = \sqrt{\frac{3a^2}{9}} = \sqrt{\left(\frac{a^2}{9} \times 3\right)} = \frac{a}{3}\sqrt{3}.$$

EXAMPLES FOR PRACTICE.

Find the value of x in each of the following equations :—

	Answers.
1. $5x + 22 = 2x + 31.$	$x = 3.$
2. $4 - 9x = 14 - 11x.$	$x = 5.$
3. $x + 18 = 3x - 5.$	$x = 11\frac{1}{2}.$
4. $x + \frac{x}{2} + \frac{x}{3} = 11.$	$x = 6.$
5. $2x - \frac{x}{2} + 1 = 5x - 2.$	$x = \frac{6}{7}.$
6. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} = \frac{7}{10}.$	$x = 1\frac{1}{5}.$
7. $\frac{x + 3}{2} + \frac{x}{3} = 4 - \frac{x - 5}{4}.$	$x = 3\frac{6}{13}.$
8. $x + a = \frac{x^2}{a + x}.$	$x = -\frac{a}{2}.$
9. $\frac{ax - b}{4} + \frac{a}{3} = \frac{bx}{2} - \frac{bx - a}{3}.$	$x = \frac{3b}{3a - 2b}.$
10. $\frac{a}{1 + x} + \frac{a}{1 - x} = b.$	$x = \frac{\sqrt{(b^2 - 2ab)}}{b}.$
11. $2 + \sqrt{3x} = \sqrt{(4 + 5x)}.$	$x = 12.$
12. $\sqrt{x} + \sqrt{(a + x)} = \frac{2a}{\sqrt{(a + x)}}.$	$x = \frac{a}{3}.$
13. $\sqrt{(a + x)} + \sqrt{(a - x)} = \sqrt{ax}.$	$x = \frac{4a^2}{a^2 + 4}.$
14. $\sqrt{a} + \sqrt{x} = \sqrt{ax}.$	$x = \frac{a}{(\sqrt{a} - 1)^2}.$
15. $\sqrt{(a + x)} + \sqrt{(a - x)} = b.$	$x = \frac{b}{2}\sqrt{(4a - b^2)}.$

	Answers.
16. $\sqrt{(a^2 + ax)} = a - \sqrt{(a^2 - ax)}$.	$x = \frac{a}{2}\sqrt{3}$.
17. $\sqrt{(x + a)} = c - \sqrt{(x + b)}$.	$x = \frac{(c^2 + b - a)^2}{4c^2} - b$.
18. $a + x = \sqrt{\{a^2 + x\sqrt{(b^2 + x^2)}\}}$.	$x = \frac{b^2}{4a} - a$.
19. $\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} + \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}} = a$	$x = \frac{a}{\sqrt{(a^2 - 4)}}$.
20. $\frac{1}{2}\sqrt{(x^2 + 3a^2)} + \frac{1}{2}\sqrt{(x^2 - 3a^2)} = x\sqrt{a}$.	$x = \left(\frac{9a^3}{4 - 4a}\right)^{\frac{1}{4}}$.
21. $\sqrt{(a^2 + x^2)} = \sqrt[4]{(b^4 + x^4)}$.	$x = \left(\frac{b^4 - a^4}{2a^2}\right)^{\frac{1}{2}}$.
22. $\left(\frac{b}{a+x}\right)^{\frac{1}{2}} + \left(\frac{c}{a-x}\right)^{\frac{1}{2}} = \left(\frac{4bc}{a^2 - x^2}\right)^{\frac{1}{4}}$.	$x = a.\frac{b + c}{b - c}$.
23. $\sqrt{(a^2 - x^2)} + x\sqrt{(a^2 - 1)} = a^2\sqrt{(1 - x^2)}$.	$x = \left(\frac{a^2 - 1}{a^2 + 3}\right)^{\frac{1}{2}}$.
24. $\sqrt[3]{(a+x)} + \sqrt[3]{(a-x)} = b$.	$x = \left\{a^2 - \left(\frac{b^3 - 2a}{3b}\right)^3\right\}^{\frac{1}{2}}$.
25. $\frac{x^2 - 1}{\frac{1}{5}(x + 1)} - \frac{5x^3 + 5}{x(x - 1) + 1} = 13 - \frac{3x - \frac{1}{4}(x + 3)}{2}$.	$x = 17$.
26. $\frac{1 + x^3}{(1 + x)^2} + \frac{1 - x^3}{(1 - x)^2} = a$.	$x = \left(\frac{a-2}{a+4}\right)^{\frac{1}{2}}$.
27. $\frac{x - 1}{\sqrt{x} + 1} = a + \frac{\sqrt{x} - 1}{2}$.	$x = (2a + 1)^2$.
28. $\sqrt{(1 + x + x^2)} + \sqrt{(1 - x + x^2)} = a$.	$x = \frac{a}{2}.\left(\frac{a^2 - 4}{a^2 - 1}\right)^{\frac{1}{2}}$.
29. $\frac{\sqrt{(a+x)} + \sqrt{(a-x)}}{\sqrt{(a+x)} - \sqrt{(a-x)}} = b$.	$x = \frac{2ab}{1 + b^2}$.
30. $(a + x)^{\frac{1}{m}} = (x^2 + nax + b^2)^{\frac{1}{2m}}$.	$x = \frac{a^2 - b^2}{(n - 2)a}$.
31. $\sqrt{(x + \sqrt{x})} - \sqrt{(x - \sqrt{x})} = a\left(\frac{x}{x + \sqrt{x}}\right)^{\frac{1}{2}}$.	$x = \left\{\frac{(a - 1)^2 + 1}{2(a - 1)}\right\}^2$.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE,

Containing two unknown quantities.

When the values of two unknown quantities are to be determined, two independent equations must necessarily be given, and when the conditions expressed by these two equations are fulfilled by the same values of the unknown quantities contained in them, these equations are *coexistent*, and are usually termed *simultaneous equations*. Any of the three following methods may be employed for the solution of such equations.

First method.—Find the value of either of the unknown quantities in terms of the other and known quantities from each equation; then put these two values equal to each other, and there will be formed an equation involving only one unknown quantity. This is the *method of comparison.*

Second method.—Find the value of either of the unknown quantities in terms of the other and known quantities from one equation, and this value substituted for it in the other equation, will give an equation involving only one unknown quantity. This is the *method of substitution.*

Third method.—Multiply or divide the two equations by such numbers or quantities as shall make the terms containing one of the unknown quantities, the same in both equations, then the difference or sum of the resulting equations, according as the equalized terms have the same or contrary signs, will give an equation containing only one unknown quantity. This is the method of *equalizing the coefficients.*

EXAMPLES.

1. Given $2x + 3y = 23$ and $5x - 2y = 10$, to find the values of x and y.

By the first method.

From these two equations, we get

$$x = \frac{23 - 3y}{2} \text{ and } x = \frac{2y + 10}{5};$$

$$\therefore \frac{23 - 3y}{2} = \frac{2y + 10}{5}, \text{ or } 115 - 15y = 4y + 20;$$

$$\text{hence } 19y = 95, \text{ and therefore } y = \frac{95}{19} = 5,$$

$$\text{and } x = \frac{2y + 10}{5} = \frac{10 + 10}{5} = \frac{20}{5} = 4.$$

By the second method.

From the first equation we get $x = \frac{23 - 3y}{2}$, and substituting this value for x in the second equation, gives

$$\frac{5(23 - 3y)}{2} - 2y = 10, \text{ or } 115 - 15y - 4y = 20;$$

$$\therefore\ 19y = 95, \text{ and hence } y = \frac{95}{19} = 5;$$

$$\text{and } x = \frac{23 - 3y}{2} = \frac{23 - 15}{2} = \frac{8}{2} = 4.$$

By the third method.

Multiply the first equation by 2, and the second by 3; then

$$\begin{aligned} 4x + 6y &= 46 \\ 15x - 6y &= 30. \end{aligned}$$

The sum of these gives $19x = 76$, or $x = 4$.

Substituting this value for x in the first equation, gives

$$8 + 3y = 23; \text{ therefore } 3y = 15, \text{ or } y = 5.$$

2. Given $\frac{2x}{5} + \frac{3y}{4} = \frac{9}{20}$, and $\frac{3x}{4} + \frac{2y}{5} = \frac{61}{120}$, to find the values of x and y.

Clearing these equations of fractions, we get

$$\begin{aligned} 8x + 15y &= 9 \quad . \ . \ . \ (1) \\ 90x + 48y &= 61 \quad . \ . \ . \ (2) \end{aligned}$$

Multiply equation (1) by 6; then we have

$$\begin{aligned} 48x + 90y &= 54 \quad . \ . \ . \ (3) \\ 90x + 48y &= 61 \quad . \ . \ . \ (2) \end{aligned}$$

Taking the sum and difference of equations (3) and (2), we get

$$\begin{aligned} 138x + 138y &= 115, \text{ or } 6x + 6y = 5 \quad . \ . \ . \ (4) \\ 42x - 42y &= 7, \text{ or } 6x - 6y = 1 \quad . \ . \ . \ (5) \end{aligned}$$

Adding and subtracting equations (4) and (5), we get

$$\begin{aligned} 12x &= 6 \quad \therefore\ x = \tfrac{1}{2} \\ 12y &= 4 \quad \therefore\ y = \tfrac{1}{3}. \end{aligned}$$

3. Given $\frac{x}{2} - 12 = \frac{y}{4} + 8$ and $\frac{x+y}{5} + \frac{x}{3} - 8 = \frac{2y-x}{4} + 27$, to find the values of x and y.

Multiply the first equation by 4, and the second by 60, to clear them of fractions; then

$$2x - 48 = y + 32,$$
$$12x + 12y + 20x - 480 = 30y - 15x + 1620.$$

Transposing and simplifying these results, we get

$$2x - \ \ y = 80 \ldots\ldots (1)$$
$$47x - 18y = 2100 \ldots (2)$$

Multiply equation (1) by 18, then

$$36x - 18y = 1440 \ldots (3)$$

Subtracting (3) from (2) gives

$$11x = 660, \therefore x = \frac{660}{11} = 60.$$

And from (1) $y = 2x - 80 = 120 - 80 = 40$.

4. Given $\frac{x+y}{3} + \frac{x-y}{4} = 59$, and $x : 3y :: 11 : 5$, to find the values of x and y.

Clearing the equation of fractions, and converting the analogy into an equation, we have the equations

$$7x + \ \ y \ = 708$$
$$5x - 33y = \ \ 0.$$

Multiply the first by 5 and the second by 7; then

$$35x + \ \ 5y \ = 3540,$$
$$35x - 231y = \ \ 0.$$

Subtracting the latter from the former, gives

$$236y = 3540, \text{ or } y = \frac{3540}{236} = 15.$$

$$\text{Hence } x = \frac{33y}{5} = \frac{33 \times 15}{5} = 33 \times 3 = 99.$$

EXAMPLES FOR PRACTICE.

Find the values of the unknown quantities in the subsequent simultaneous equations:—

	Answers.
1. $x + y = a$, and $x - y = b$.	$x = \frac{a+b}{2}, y = \frac{a-b}{2}$.
2. $5x - 3y = 9$, and $2x + 5y = 16$.	$x = 3, y = 2$.
3. $4x + y = 34$, and $4y + x = 16$.	$x = 8, y = 2$.
4. $2x + 3y = 16$, and $3x - 2y = 11$.	$x = 5, y = 2$.
5. $ax + by = c$, and $px + qy = r$.	$x = \frac{cq - br}{aq - bp}, y = \frac{ar - cp}{aq - bp}$.
6. $\frac{x}{9} + \frac{y}{8} = 26$, and $\frac{x}{8} + \frac{y}{9} = 25$.	$x = 72, y = 144$.
7. $\frac{5}{x} + \frac{3}{y} = \frac{17}{15}$, and $\frac{18}{x} - \frac{20}{y} = 1$.	$x = 6, y = 10$.
8. $\frac{a}{x} + \frac{b}{y} = c$, and $\frac{b}{x} + \frac{a}{y} = d$.	$x = \frac{a^2 - b^2}{ac - bd}, y = \frac{a^2 - b^2}{ad - bc}$.
9. $\frac{1}{3x} + \frac{1}{5y} = \frac{3}{5}$, and $\frac{1}{5x} + \frac{1}{3y} = \frac{1}{5}$.	$x = \frac{4}{9}, y = -\frac{4}{3}$.
10. $\frac{x}{7} + 7y = 99$, and $\frac{y}{7} + 7x = 51$.	$x = 7, y = 14$.
11. $2x - \frac{y-3}{5} = 4$, and $3y + \frac{x-2}{3} = 9$.	$x = 2, y = 3$.
12. $x - \frac{y-2}{7} = 5$, and $4y - \frac{x+10}{3} = 3$.	$x = 5, y = 2$.
13. $\left.\begin{aligned} &\frac{7+x}{5} - \frac{2x-y}{4} = 3y - 5, \\ &\frac{5y-7}{2} + \frac{4x-3}{6} = 18 - 5x. \end{aligned}\right\}$	$x = 3, y = 2$.
14. $\left.\begin{aligned} &\frac{x-2}{5} - \frac{10-x}{3} = \frac{y-10}{4}, \\ &\frac{2y+4}{3} - \frac{2x+y}{8} = \frac{x+13}{4}. \end{aligned}\right\}$	$x = 7, y = 10$.

Answers.

15. $$\left.\begin{aligned}\frac{3x+4y+3}{10}-\frac{3x-y}{15}&=5+\frac{y-8}{5},\\ \frac{9y+5x-8}{12}-\frac{x+y}{4}&=\frac{7x+6}{11}.\end{aligned}\right\}\qquad x=7,\ y=9.$$

16. $$\left.\begin{aligned}\frac{3x-2y}{3}+\frac{11y-2}{8}&=\frac{4x-3y+5}{7}+\frac{45-x}{5},\\ 45-\frac{4x-2}{3}&=\frac{55x+71y+1}{18}.\end{aligned}\right\}\qquad x=5,\ y=6.$$

17. $$\left.\begin{aligned}\frac{x}{4}+\frac{y}{6}&=1-\frac{y}{8},\\ \frac{y}{4}+\frac{x}{6}&=1+\frac{y}{8}.\end{aligned}\right\}\qquad x=9\tfrac{3}{5},\ y=-4\tfrac{4}{5}.$$

The following examples may be solved in a similar manner, though the equations are not of the first degree.

Answers.

18. $x+y=a$, and $x^2-y^2=b$. $\quad x=\dfrac{a^2+b}{2a}, y=\dfrac{a^2-b}{2a}.$

19. $x^2+y^2=c$, and $x:y::a:b$. $\quad x=a\left(\dfrac{c}{a^2+b^2}\right)^{\frac{1}{2}}, y=b\left(\dfrac{c}{a^2+b^2}\right)^{\frac{1}{2}}.$

20. $x^2-y^2=c$, and $x+y:a::x-y:b$. $\quad x=\dfrac{a+b}{2}\left(\dfrac{c}{ab}\right)^{\frac{1}{2}}, y=\dfrac{a-b}{2}\left(\dfrac{c}{ab}\right)^{\frac{1}{2}}.$

21. $x^2+y^2=a$, and $x^2-y^2=b$. $\quad x=\left(\dfrac{a+b}{2}\right)^{\frac{1}{2}}, y=\left(\dfrac{a-b}{2}\right)^{\frac{1}{2}}.$

22. $x^2+xy=a$, and $y^2+xy=b$. $\quad x=\dfrac{a}{\surd(a+b)}, y=\dfrac{b}{\surd(a+b)}.$

23. $y^2+y^{-2}=x^2+2$, and $y^2+xy=a$. $\quad x=\dfrac{a-1}{\{2(a+1)\}^{\frac{1}{2}}}, y=\left(\dfrac{a+1}{2}\right)^{\frac{1}{2}}.$

24. $$\left.\begin{aligned}a(x^2+y^2)-b(x^2-y^2)&=2a\\ (a^2-b^2)(x^2-y^2)&=4ab\end{aligned}\right\}\qquad x=\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}, y=\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}}.$$

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE,

Containing three or more unknown quantities.

When there are three unknown quantities, there must be three independent equations, and from these three equations, an equation may be found containing only one unknown quantity, by the following

Rule.—Find the values of one of the unknown quantities, in each of the three given equations, as if all the rest were known; then put the first of these values equal to the second, and either the first or second equal to the third, and there will arise two new equations with only two unknown quantities in them, the values of which may be found as in the former case; and thence the value of the third.

Or, multiply each of the equations by such numbers or quantities as will make one of their terms the same in each; then, having *subtracted* separately each of these resulting equations from one another, or from the third, in succession, when the signs are *like;* or added them together, when the signs are unlike, as the case may require, there will remain two equations only, which may be resolved similarly, or by any of the former rules.

And in nearly the same way may four, five, &c. unknown quantities be found from the same number of independent simple equations; but, in cases of this kind, there are frequently shorter and more commodious methods of operation, which can only be acquired by practice.

EXAMPLES.

1. Given $x + y + z = 29$, $x + 2y + 3z = 62$, and $\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 10$, to find the values of x, y, and z.

Subtract the first equation from the second, then

$$y + 2z = 33 \ldots\ldots. (1)$$

From the first of the given equations, subtract the double of the third, then

$$\frac{y}{3} + \frac{z}{2} = 9, \text{ or } 2y + 3z = 54 \ldots\ldots. (2)$$

From twice equation (1) subtract equation (2), then we get $z = 12$; hence

$$y = 33 - 2z = 33 - 24 = 9, \text{ and}$$
$$x = 29 - y - z = 29 - 9 - 12 = 8.$$

2. Given $2x + 4y - 3z = 22$, $4x - 2y + 5z = 18$, and $6x + 7y - z = 63$, to find the values of x, y, and z.

Multiply the first equation by 2, and from the result subtract the second, then

$$10y - 11z = 26 \ldots\ldots (a)$$

From three times the first equation subtract the third; then

$$5y - 8z = 3 \ldots\ldots (b)$$

From equation (a) subtract twice equation (b), then

$$5z = 20, \text{ and } z = 4;$$

hence from (b) $5y = 8z + 3 = 35$, and $y = 7$;
therefore $2x = 22 - 4y + 3z = 4$, and $x = 2$.

Find the values of x, y, and z in the following simultaneous equations:—

1.	Ans.	2.	Ans.
$x + y + z = 53$	$x = 24$	$3x + 2y - z = 20$	$x = 5$
$x + 2y + 3z = 105$	$y = 6$	$2x + 3y + 6z = 70$	$y = 6$
$x + 3y + 4z = 134$	$z = 23$	$x - y + 6z = 41$	$z = 7$

3.		4.	
$7x + 5y + 2z = 79$	$x = 4$	$x + \frac{1}{2}y + \frac{1}{3}z = 32$	$x = 12$
$8x + 7y + 9z = 122$	$y = 9$	$\frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z = 15$	$y = 20$
$x + 4y + 5z = 55$	$z = 3$	$\frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z = 12$	$z = 30$

5.		6.	
$5x + 3y = 65$	$x = 7$	$7x - 4z = 4$	$x = 4$
$2y - z = 11$	$y = 10$	$4x - 3y = 1$	$y = 5$
$3x + 4z = 57$	$z = 9$	$8z - 9y = 3$	$z = 6$

7.		8.	
$\frac{3}{x} - \frac{4}{5y} + \frac{1}{z} = \frac{38}{5}$	$x = \frac{1}{2}$	$\frac{5}{x} - \frac{4}{y} + \frac{2}{z} = 48$	$x = \frac{1}{10}$
$\frac{1}{3x} + \frac{1}{2y} + \frac{2}{z} = \frac{61}{6}$	$y = \frac{1}{3}$	$\frac{3}{x} + \frac{3}{y} - \frac{4}{z} = 24$	$y = \frac{1}{2}$
$\frac{4}{5x} - \frac{1}{2y} + \frac{4}{z} = \frac{161}{10}$	$z = \frac{1}{4}$	$\frac{2}{x} - \frac{5}{y} + \frac{3}{z} = 19$	$z = \frac{1}{3}$

9.	Ans.	10.	Ans.
$x + y = a$	$x = \frac{a + b - c}{2}$	$\frac{1}{x} + \frac{1}{y} = a$	$x = \frac{2}{a + b - c}$
$x + z = b$	$y = \frac{a - b + c}{2}$	$\frac{1}{x} + \frac{1}{z} = b$	$y = \frac{2}{a - b + c}$
$y + z = c$	$z = \frac{-a + b + c}{2}$	$\frac{1}{y} + \frac{1}{z} = c.$	$z = \frac{2}{-a + b + c}.$

11.		12.	
$\frac{xy}{x + y} = 3$	$x = 4\frac{4}{5}$	$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{11}{12}.$	$x = 2$
$\frac{xz}{x + z} = 4$	$y = 8$	$-\frac{3}{y} + \frac{5}{x} = \frac{7}{4}$	$y = 4$
$\frac{yz}{y + z} = 6$	$z = 24.$	$\frac{3}{y} - \frac{4}{z} = \frac{1}{12}.$	$z = 6.$

PROBLEMS PRODUCING SIMPLE EQUATIONS.

The usual method of resolving algebraic questions is first to denote the quantities that are to be found by x, y, or some of the other final letters of the alphabet; then, having properly examined the state of the question, perform with these letters, and the known quantities, by means of the common signs, the same operations and reasonings that it would be necessary to make if the quantities were known, and it was required to verify them; the conclusion derived from the solution of the equation thus formed will give the result sought.

Or, it is generally best, when it can be done, to denote only one of the unknown quantities by x, y, or z; and then to determine the expression for the others from the nature of the question in terms of it; after which the same method of reasoning may be followed as above. And, in certain cases, the substituting for the sums and differences of quantities, or other methods of proceeding, may be used; which experience alone can suggest.

1. What number is that to which if 12 be added, one-fourth of the sum is equal to 17?

Let x represent the number; then adding 12 to it we get $x + 12$, and one-fourth of this is $\frac{1}{4}(x + 12)$; hence we have the equation

$\frac{1}{4}(x + 12) = 17$, or $x + 12 = 68$; hence $x = 56 =$ the number required.

2. What number is that whose third part exceeds its fourth part by 16?

Let x denote the number; then one-third of it is $\frac{x}{3}$ and one-fourth is $\frac{x}{4}$; hence

$$\frac{x}{3} - \frac{x}{4} = 16. \therefore 4x - 3x = 192, \text{ and } x = 192.$$

3. Find two numbers whose sum is 40, and difference 16.

Let $x =$ the less number; then $x + 16 =$ the greater number, and $x + x + 16 = 40$, or $2x = 24$; therefore $x = 12$, the less number, and $x + 16 = 12 + 16 = 28$, the greater number.

4. Divide L.1000 among A, B, C, so that A shall have L.72 more than B, and C L.100 more than A.

Let $x =$ the share of B; then $x + 72 =$ the share of A, and $x + 172 =$ the share of C; then the sum of these shares is $3x + 244$; and therefore

$$\begin{array}{lrcl} & 3x + 244 = 1000, \text{ and } x = & \text{L.}252 & = \text{B's share.} \\ \text{Hence} & x + 72 = 252 + 72 = & 324 & = \text{A's share} \\ \text{and} & x + 172 = 252 + 172 = & 424 & = \text{C's share} \\ \hline & & \text{L.}1000 & = \text{sum.} \end{array}$$

5. Divide L.1000 between two persons, so that their shares shall be as 7 to 9.

Let $7x$ and $9x$ represent their shares, and they will obviously be as 7 to 9; hence

$$7x + 9x = 1000, \text{ or } 16x = 1000, \therefore x = \text{L.}62, 10\text{s.}$$

$$\begin{array}{lrl} \text{Hence } 7x = & \text{L.}62, 10\text{s.} \times 7 = \text{L.}437, 10\text{s.} & = \text{share of one} \\ 9x = & \text{L.}62, 10\text{s.} \times 9 = \text{L.}562, 10\text{s.} & = \text{share of the other} \\ \hline & \text{L.}1000 & = \text{sum.} \end{array}$$

6. The paving of a square court with stones, at 2s. a-yard, cost as much as the inclosing it with palisades at 5s. a-yard; find the length of the side of the square.

Let $x =$ the length of the side of the square in yards; then
$4x =$ number of yards of inclosure
$x^2 =$ number of square yards of pavement,
$4x \times 5 = 20x =$ price of inclosing in shillings
$x^2 \times 2 = 2x^2 =$ price of paving in shillings; hence

$$2x^2 = 20x, \text{ or } 2x = 20, \text{ and } x = 10 \text{ yards.}$$

7. From a cask of wine, which had leaked a third part, 21 gallons were drawn, and the cask being then guaged was found half full; how many gallons did it originally contain?

Let $x =$ the number of gallons in the cask originally; then $\frac{1}{3}x$ gallons were lost by leakage, and 21 gallons were drawn. Now these together must be equal to half the number of gallons in the cask at first; hence

$$\frac{x}{3} + 21 = \frac{x}{2}, \text{ or } 2x + 126 = 3x \quad \therefore x = 126 \text{ gallons.}$$

8. What fraction is that whose value is $\frac{1}{3}$ when its numerator is increased by 1, and whose value is $\frac{1}{4}$ when its denominator is increased by 1.

Let $\frac{x}{y} =$ the fraction; then we have

$$\frac{x+1}{y} = \tfrac{1}{3}, \text{ or } 3x + 3 = y, \text{ or } y = 3x + 3,$$

$$\frac{x}{y+1} = \tfrac{1}{4}, \text{ or } 4x = y + 1, \text{ or } y = 4x - 1;$$

$$\therefore 4x - 1 = 3x + 3, \quad \text{and} \quad x = 4;$$

$$\therefore \quad y = 3x + 3 = 12 + 3 = 15,$$

$$\text{and } \frac{x}{y} = \tfrac{4}{15} = \text{the fraction required.}$$

9. A market woman bought a certain number of oranges at 2 a-penny, and an equal number at 3 a-penny; but having sold the whole at the rate of 5 for 2d., found she had lost 4d.: how many oranges were bought?

Let $x =$ the number of oranges of each kind; then $\frac{x}{2} =$ the buying price of one of the first kind, and $\frac{x}{3} =$ the price of one of the second kind. Now the whole number of oranges is $2x$, and hence

$$5 : 2x :: 2 : \frac{4x}{5} = \text{selling price of the whole at 5 for 2d.};$$

$$\therefore \frac{x}{2} + \frac{x}{3} = \frac{4x}{5} + 4, \text{ or } 15x + 10x = 24x + 120;$$

hence $x = 120 =$ number of oranges of each kind.

10. If A can do a piece of work in 10 days, and B the same in 13 days; in what time will A and B together finish it?

Let x = the time required; then

A can do $\frac{1}{10}$ in a day, and $\frac{x}{10}$ in x days,

B can do $\frac{1}{13}$ in a day, and $\frac{x}{13}$ in x days;

but A and B together can finish the piece of work in x days; hence

$$\frac{x}{10} + \frac{x}{13} = 1, \text{ or } 13x + 10x = 130, \therefore x = \frac{130}{23} = 5\tfrac{15}{23} \text{ days.}$$

11. If one agent, A, alone can produce an effect e in the time a, and another agent, B, alone in the time b; in what time will both together produce the same effect?

Let x = the time sought; then

A can do $\frac{e}{a}$ part in 1 day, and $\frac{ex}{a}$ part in x days,

B can do $\frac{e}{b}$ part in 1 day, and $\frac{ex}{b}$ part in x days;

$$\therefore \frac{ex}{b} + \frac{ex}{b} = e, \text{ or } \frac{x}{a} + \frac{x}{b} = 1, \therefore ax + bx = ab;$$

hence $x = \frac{ab}{a+b}$ = the time required.

12. How much rye at 4s. 6d. a bushel must be mixed with 50 bushels of wheat at 6s. a bushel, that the mixture may be worth 5s. per bushel?

Let x = number of bushels of rye; then

$9x$ = price of x bushels of rye in sixpences, and

600 = price of 50 bushels of wheat in sixpences.

But $(50+x) \times 10 = 500 + 10x$ = price of mixture in sixpences; hence

$$9x + 600 = 500 + 10x, \text{ and } x = 100 \text{ bushels of rye.}$$

13. A labourer engaged to serve for 40 days, on condition that for every day he worked he should receive 20d., but for every day he was absent he should forfeit 8d. Now at the end of the time he had to receive L.1, 11s. 8d.; how many days did he work, and how many was he absent?

Let x = the number of days he was absent; then

$40 - x$ = the number of days he worked,

$8x$ = the sum forfeited by absence,

$(40 - x) \times 20 = 800 - 20x$ = sum earned; therefore

$800 - 20x - 8x$ = L.1, 11s. 8d. = 380d.;

$\therefore 28x = 800 - 380 = 420$, and $x = 15$ days absent;

hence $40 - 15 = 25$ = number of days he worked.

14. A garrison of 500 men was victualled for 48 days; but after 15 days it was reinforced, and then the provisions were exhausted in 11 days; find the number of men in the reinforcement.

Let $x =$ the number of men in the reinforcement; then as the consumption is proportional to the product of the men by the time, we have

$500 \times 48 = 500 \times 15 + (500 + x) \times 11$; hence

$11x + 5500 = 500 \times (48 - 15) = 500 \times 33 = 16500$;

$\therefore 11x = 11000$, and $x = 1000 =$ the number of men in the reinforcement.

QUESTIONS FOR PRACTICE.

1. It is required to divide a line of 14 inches in length into two such parts, that one of them may be three-fourths of the other.

Ans. 8 and 6.

2. My purse and money together are worth 20s., and the money is worth 7 times as much as the purse; how much is there in it?

Ans. 17s. 6d.

3. A shepherd, being asked how many sheep he had in his flock, said, If I had as many more, half as many more, and 7 sheep and a half, I should have just 500; how many had he? Ans. 197.

4. A post is one-fourth of its length in the mud, one-third in the water, and 10 feet above the water; what is its whole length?

Ans. 24 feet.

5. After paying away a fourth of my money, and then a fifth of the remainder, I had 72 guineas left; what had I at first?

Ans. 120 guineas.

6. It is required to divide L.300 between A, B, and C, so that A may have twice as much as B, and C as much as A and B together.

Ans. A L.100, B L.50, and C L.150.

7. A person, at the time he was married, was 3 times as old as his wife; but after they had lived together 15 years, he was only twice as old; what were their ages on their wedding-day?

Ans. Bride's age, 15, Bridegroom's, 45.

8. It is required to find a number such, that if 5 be subtracted from it, two-thirds of the remainder shall be 40? Ans. 65.

9. At a certain election, 1296 persons voted, and the successful candidate had a majority of 120; how many voted for each?

Ans. 708 for one, and 588 for the other.

10. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140; what is the age of each?

Ans. A's 84, B's 42, and C's 14.

11. Two persons, A and B, lay out equal sums of money in trade; A gains L.126, and B loses L.87, and A's money is now double of B's; what did each lay out? Ans. L.300.

12. A person bought a chaise, horse, and harness, for L.60; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness; what did he give for each?

Ans. L.13, 6s. 8d. for the horse, L.6, 13s. 4d. for the harness, and L.40 for the chaise.

13. A person was desirous of giving 3d. a-piece to some beggars, but found he had not money enough in his pocket by 8d, he therefore gave them each 2d., and had then 3d. remaining; required the number of beggars? Ans. 11.

14. A servant agreed to live with his master for L.8 a-year and a livery, but was turned away at the end of seven months, and received only L.2, 13s. 4d. and his livery; what was its value.

Ans. L.4, 16s.

15. A person left L.560 between his son and daughter, in such a manner, that for every half-crown the son should have, the daughter was to have a shilling; what were their respective shares?

Ans. Son L.400, Daughter L.160.

16. Two persons, A and B, have both the same income; A saves a fifth of his yearly, but B, by spending L.50 per annum more than A, at the end of four years finds himself L.100 in debt; what was their income?

Ans. L.125.

17. A person at a tavern borrowed as much money as he had about him, and out of the whole spent 1s.; he then went to a second tavern, where he also borrowed as much as he had now about him, and out of the whole spent 1s.; and going on, in this manner, to a third and fourth tavern, he found, after spending his shilling at the latter, that he had nothing left; how much money had he at first? Ans. 11¼d.

18. It is required to divide the number 75 into two such parts, that three times the greater shall exceed seven times the less by 15.

Ans. 54 and 21.

19. In a mixture of British spirits and water, one half of the whole plus 25 gallons was spirits, and a third part minus 5 gallons was water; how many gallons were there of each?

Ans. 85 of spirits, and 35 of water.

20. Two travellers set out at the same time from London and York, whose distance from each other is 197 miles; one of them goes 14 miles a day, and the other 16; in what time will they meet?

Ans. 6 days $13\frac{3}{5}$ hours.

21. There is a fish whose tail weighs 9 lb., his head weighs as much as his tail and half his body, and his body weighs as much as his head and his tail; what is the whole weight of the fish? Ans. 72 lb.

22. It is required to divide the number 10 into three such parts, that if the first be multiplied by 2, the second by 3, and the third by 4, the three products shall be all equal. Ans. $4\frac{8}{13}$, $3\frac{1}{13}$, and $2\frac{4}{13}$.

23. It is required to divide the number 36 into three such parts, that half the first, a third of the second, and a fourth of the third, shall be all equal to each other. Ans. The parts are 8, 12, and 16.

24. A person in play lost a fourth of his money, and then won back 3s., after which he lost a third of what he now had, and then won back 2s.; lastly, he lost a seventh of what he then had, and after this found he had but 12s. remaining; what had he at first? Ans. 20s.

25. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3, but two of the greyhound's leaps are as much as three of the hare's; how many leaps must the greyhound take to catch the hare?

Ans. 300.

26. It is required to divide the number 90 into four such parts, that if the first part be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient, shall be all equal.

Ans. The parts are 18, 22, 10, and 40.

27. A person, after spending L.10 more than a third of his yearly income, found he had L.15 more than half of it remaining; what was his income? Ans. L.150.

28. A man and his wife usually drank out a cask of beer in 12 days, but when the man was from home, it lasted the woman 30 days; how many days would the man alone be in drinking it?

Ans. 20 days.

29. A general, ranging a division of his army in the form of a solid square, finds he has 34 men to spare, but increasing the side by one man, he wants 59 to fill up the square; of how many soldiers did the division consist? Ans. 2150.

30. A garrison is victualled for 30 days; after 10 days it was reinforced by 3000 men, and then the provisions lasted only 5 days; find the original number of men in the garrison. Ans. 1000 men.

31. A waterman rowed down a river and up again in 6 hours, and found that he could row 9 miles an-hour with the stream, and only 3 miles an-hour against it; how far did he row? Ans. 13½ miles.

32. A garrison of a soldiers provisioned for m months, at the rate of p ounces per man in a day, expects to be reinforced by b men at the end of n months; what must be the uniform daily allowance that the provisions may just be consumed at the expiration of the m months?

Ans. $\dfrac{mpa}{ma + (m - n)b}$ ounces.

33. How many lines are contained in a page of a book, and how many letters at an average in each line, if one line added to the page, and one letter to each line, increase the number of letters in the page by 96; but if two lines be added to the original page and four letters to each line, the number of letters will be increased by 286.

Ans. 44 lines and 51 letters in a line.

34. At the review of an army, the troops were drawn up into a solid mass, 40 deep, and one-fourth as many men in front as there were spectators. But if the depth had been increased by 5 men, and the spectators drawn up in the mass with the army, the number of men in front would have been 100 fewer than before; find the number of men in the army, and the number of spectators.

Ans. Army consisted of 180,000 men,
and there were 18,000 spectators.

35. Two persons, A and B, working together can finish a certain work in 12 days; but after working together for 6 days, A was withdrawn and B left to complete the work, which he accomplished in 18 days; find in what time each could finish the work alone.

Ans. A in 18 days, and B in 36 days.

36. A bill of 25 guineas was paid with crowns and half-guineas, and twice the number of half-guineas exceeded three times the number of crowns by 17, how many were there of each.

Ans. 40 half-guineas, and 21 crowns.

37. A rectangular bowling-green contains a certain number of square feet; and if it were 5 feet broader and 4 feet longer, it would contain 116 feet more; but if it were 4 feet broader and 5 feet longer, it would contain 113 feet more; find its length and breadth, and the number of square feet it contains.

Ans. Length 12 feet, breadth 9, and content 108 sq. feet.

38. A person has 55 coins, consisting of crowns and shillings, and their amount is 143 shillings: how many has he of each kind of coin.

Ans. 22 crowns and 33 shillings.

39. If A and B can do a piece of work in m days, A and C in n days, B and C in r days; what is the proportion of the amount of work done by A, B, and C respectively in a day?

Ans. $\text{A's} = \dfrac{mr + nr - mn}{2mnr}$, $\text{B's} = \dfrac{mn + nr - mr}{2mnr}$

and $\text{C's} = \dfrac{mn + mr - nr}{2mnr}$.

QUADRATIC EQUATIONS.

A *quadratic equation* is one which contains the square of the unknown quantity, and is either pure or adfected.

A *pure* quadratic contains only the square of the unknown quantity, as $x^2 = 25$, and $ax^2 = b$.

An *adfected* quadratic contains both the first and second powers of the unknown quantity, as $x^2 - 3x = 2$, and $ax^2 + bx = c$.

Pure quadratics are solved, as in simple equations, by collecting the unknown quantities on one side, and the known quantities on the other; then taking the square root of both sides, and prefixing the double sign (±) to the second side. Thus if $ax^2 = b$, then $x^2 = \frac{b}{a}$, and $x = \pm \sqrt{\frac{b}{a}} = \pm \frac{1}{a}\sqrt{ab}$; hence pure quadratics have two equal roots with contrary signs.

Affected quadratics may be reduced to either of the forms

$$x^2 + ax = b \quad . \quad . \quad . \quad (1) \quad \text{or, } ax^2 + bx = c \quad . \quad . \quad . \quad (2).$$

Since the square of a binomial quantity consists of three terms, it is obvious that a term must be added to the left member of (1) to make a complete square. Let m^2 be the term to be added, then if $x^2 + ax + m^2$ be a complete square, its root will be found thus :—

$$\begin{array}{r|l} & x^2 + ax + m^2 \big(x + \frac{a}{2} \\ & x^2 \\ \cline{2-2} 2x + \frac{a}{2} \big) & \quad ax + m^2 \\ & \quad ax + \frac{1}{4}a^2 \\ \cline{2-2} \end{array}$$

Hence we see that when $x^2 + ax + m^2$ is a complete square, we must have the condition

$$m^2 = \tfrac{1}{4}a^2 = (\tfrac{1}{2}a)^2, \text{ or } m = \tfrac{1}{2}a.$$

We have then the following rule for the solution of a quadratic of the form $x^2 + ax = b$.

Rule.—Transpose all the terms involving the unknown quantity to one side of the equation, and the known terms to the other, and let the term which contains the square of the unknown quantity be positive, and stand first in the equation. And if the square of the unknown has a coefficient, let all the terms be divided by it.

Then to each side, add the square of half the coefficient of the first or simple power of the unknown quantity, and the first side will be a complete square. Extract the square root of each side, prefix the double sign (±) to the second, and the two values of the unknown quantity will be readily obtained.

Thus, if $\qquad x^2 + ax = b$, be the given equation,

then, $\qquad x^2 + ax + \frac{a^2}{4} = \frac{a^2}{4} + b,$

and extracting the root, $x + \frac{a}{2} = \pm \sqrt{\left(\frac{a^2}{4} + b\right)}$;

$\therefore$ $x = -\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + b\right)}$.

If $x^2 - ax = b$, be the equation, then $x = \frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + b\right)}$;

or if $x^2 - ax = -b$; then $x = \frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - b\right)}$.

If the proposed equation be of the form $ax^2 + bx = c$, then multiplying all the terms by $4a$ gives $4a^2x^2 + 4abx = 4ac$, where the first term $4a^2x^2$ is a square. Now if m^2 be the term to be added to $4a^2x^2 + 4abx$ to make a complete square; then the root of $4a^2x^2 + 4abx + m^2$ will be obtained thus :—

$$\begin{array}{l} 4a^2x^2 + 4abx + m^2(2ax + b \\ 4a^2x^2 \\ \hline 4ax + b)\ 4abx + m^2 \\ \qquad\qquad 4abx + b^2 \\ \hline \end{array}$$

Hence we see that for $4a^2x^2 + 4abx + m^2$ to be a complete square, we must have $m^2 = b^2$, or $m = b$. We have therefore the following rule for the solution of quadratics of the form $ax^2 + bx = c$.

Rule.—Multiply all the terms of the equation by four times the coefficient of the square of the unknown quantity, and to each side of the result add the square of the coefficient of the simple power of the unknown. Extract the square root of both sides; prefix the double sign to the second, and the values of the roots will be obtained as before.

Thus if $ax^2 + bx = c$ be the equation,
then $4a^2x^2 + 4abx = 4ac$, by multiplying by $4a$,
and $4a^2x^2 + 4abx + b^2 = b^2 + 4ac$, by adding b^2 to both sides,
$\therefore$ $2ax + b = \pm \sqrt{(b^2 + 4ac)}$,
and $x = \frac{-b \pm \sqrt{(b^2 + 4ac)}}{2a}$.

This method avoids fractions in the operation, and may be frequently used with advantage.

Note.—Equations of the form $x^{2m} + ax^{m} = b$, or $x^{m} + ax^{\frac{m}{2}} = b$, where the index of the unknown quantity in one of the terms is twice that in the other, are resolved like quadratics; thus

$$x^{2m} + ax^{m} = b, \qquad x^{m} + ax^{\frac{m}{2}} = b,$$

$$x^{2m} + ax^{m} + \frac{a^2}{4} = \frac{a^2}{4} + b, \qquad x^{m} + ax^{\frac{m}{2}} + \frac{a^2}{4} = \frac{a^2}{4} + b;$$

$$\therefore\ x^{m} + \frac{a}{2} = \pm\sqrt{\left(\frac{a^2}{4} + b\right)}, \qquad \therefore\ x^{\frac{m}{2}} + \frac{a}{2} = \pm\sqrt{\left(\frac{a^2}{4} + b\right)},$$

$$\text{or, } x^{m} = -\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + b\right)}, \qquad \text{or, } x^{\frac{m}{2}} = -\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + b\right)},$$

$$\therefore\ x = \left\{-\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + b\right)}\right\}^{\frac{1}{m}} \quad \therefore\ x = \left\{-\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + b\right)}\right\}^{\frac{2}{m}}.$$

EXAMPLES.

1. Given $x^2 + 4x = 140$, to find the value of x.

Adding the square of half the coefficient of x, viz. 2^2 to both sides, we get

$$x^2 + 4x + 4 = 144,$$

its root is $\quad x + 2 = \pm 12,$

$$\left.\begin{aligned} \therefore x &= -2 + 12 = 10 \\ \text{or } x &= -2 - 12 = -14 \end{aligned}\right\}\text{Ans.}$$

Both these values of x, when substituted for it in the proposed equation, verify the equation, and are therefore the two roots of the equation.

2. Given $x^2 - 12x + 30 = 3$, to find the value of x.

By transposition, $x^2 - 12x = -27$;

$\therefore\ x^2 - 12x + 6^2 = 36 - 27 = 9;$

$\therefore\qquad x - 6 = \pm 3$

or $\qquad x = 6 \pm 3 = 9$ or $x = 3$.

3. Given $3x^2 - 3x + 6 = 5\frac{1}{3}$, to find the value of x.

Transposing and dividing by 3 gives the equation

$$x^2 - x = -\tfrac{2}{9}.$$

Here the coefficient of x is 1, and the square of half this coefficient is $\frac{1}{4}$; hence

$$x^2 - x + \tfrac{1}{4} = \tfrac{1}{4} - \tfrac{2}{9} = \tfrac{1}{36},$$

$$\therefore \quad x - \tfrac{1}{2} = \pm \tfrac{1}{6},$$

$$\text{and} \quad x = \tfrac{1}{2} \pm \tfrac{1}{6} = \tfrac{2}{3} \text{ or } x = \tfrac{1}{3}.$$

4. Given $3x^2 + 2x = 85$, to find the values of x.

Multiplying all the terms by 12, and adding the square of 2 to both sides, gives

$$36x^2 + 24x + 4 = 1024;$$

$$\text{its root is} \quad 6x + 2 = \pm 32;$$

$$\therefore 6x = -2 \pm 32 = 30 \text{ or } -34;$$

$$\text{hence } x = 5, \text{ or } x = -\tfrac{17}{3}.$$

5. Given $\frac{1}{2}x^6 - \frac{1}{4}x^3 = -\frac{1}{32}$, to find the value of x.

Multiplying by 2, gives $x^6 - \frac{1}{2}x^3 = -\frac{1}{16}$; hence

$$x^6 - \tfrac{1}{2}x^3 + \tfrac{1}{16} = 0;$$

$$\therefore x^3 - \tfrac{1}{4} = 0, \text{ or } x^3 = \tfrac{1}{4};$$

$$\text{hence } x = \sqrt[3]{\tfrac{1}{4}} = \sqrt[3]{\tfrac{2}{8}} = \tfrac{1}{2}\sqrt[3]{2}.$$

6. Given $2x^{\frac{2}{3}} + 3x^{\frac{1}{3}} = 2$, to find the values of x.

Dividing by 2, gives $x^{\frac{2}{3}} + \frac{3}{2}x^{\frac{1}{3}} = 1$, hence

$$x^{\frac{2}{3}} + \tfrac{3}{2}x^{\frac{1}{3}} + \tfrac{9}{16} = \tfrac{9}{16} + 1 = \tfrac{25}{16};$$

$$\therefore x^{\frac{1}{3}} + \tfrac{3}{4} = \pm \tfrac{5}{4};$$

$$\text{hence } x^{\frac{1}{3}} = \frac{-3 \pm 5}{4} = \tfrac{1}{2} \text{ or } -2, \therefore x = \tfrac{1}{8} \text{ or } x = -8.$$

7. Given $x^4 - 12x^3 + 44x^2 - 48x = 9009$, to find the values of x.

This equation is the same as

$$x^4 - 12x^3 + 36x^2 + 8\,(x^2 - 6\,x) = 9009,$$

$$\text{or} \qquad (x^2 - 6x)^2 + 8\,(x^2 - 6x) = 9009;$$

$$\therefore (x^2 - 6x)^2 + 8\,(x^2 - 6x) + 16 = 9025;$$

$$\therefore x^2 - 6x + 4 = \pm\, 95,$$

$$\text{or } x^2 - 6x + 9 = 100 \text{ or } -90.$$

$$\begin{array}{ll} \text{Whence } x - 3 = \pm\, 10, & \text{or } x - 3 = \pm\, 3\sqrt{-10} \\ \therefore x = 3 + 10 = 13, & x = 3 + 3\sqrt{-10} \\ & \text{or} \\ x = 3 - 10 = -7 & x = 3 - 3\sqrt{-10}, \end{array}$$

which are the four roots of the biquadratic equation proposed.

8. Given $x - 1 = \dfrac{6}{\sqrt{x}}$, to find the values of x.

Here $x - 1 = \dfrac{6}{\sqrt{x}} = \dfrac{6\sqrt{x}}{x}$, or $x^2 - x = 6\sqrt{x}$.

Adding $4x$ to both sides, $\quad x^2 + 3x = 4x + 6\sqrt{x}$,

completing the square, $\quad x^2 + 3x + \frac{9}{4} = 4x + 6\sqrt{x} + \frac{9}{4}$.

Extracting the root, $\quad x + \frac{3}{2} = \pm\,(2\sqrt{x} + \frac{3}{2})$.

From the equation $x + \frac{3}{2} = 2\sqrt{x} + \frac{3}{2}$, we get $x = 2\sqrt{x}$, or $\sqrt{x} = 2$; hence $x = 4$; and from the equation $x + \frac{3}{2} = -2\sqrt{x} - \frac{3}{2}$, we have

$$x + 2\sqrt{x} = -3,$$

$$\therefore x + 2\sqrt{x} + 1 = -2,$$

$$\text{or } \sqrt{x} + 1 = \pm\sqrt{-2}.$$

Hence $\sqrt{x} = -1 \pm \sqrt{-2}$, and $x = -1 \pm 2\sqrt{-2}$.

9. Given $5x - 7x^2 + 8\sqrt{(7x^2 - 5x + 1)} = 8$, to find x.

Changing the signs of all the terms, and adding 1 to both sides, gives

$$7x^2 - 5x + 1 \; - 8\,(7x^2 - 5x + 1)^{\frac{1}{2}} = -8 + 1 = -7;$$

$$\therefore (7x^2 - 5x + 1) - 8\,(7x^2 - 5x + 1)^{\frac{1}{2}} + 16 = 9;$$

$$\therefore (7x^2 - 5x + 1)^{\frac{1}{2}} - 4 = \pm 3.$$

Hence $(7x^2 - 5x + 1)^{\frac{1}{2}} = 7$, or $(7x^2 - 5x + 1)^{\frac{1}{2}} = 1$,

$$\therefore 7x^2 - 5x + 1 = 49, \text{ or } 7x^2 - 5x + 1 = 1.$$

Resolving these two quadratics, we get from the first

$$x = 3 \text{ or } -\tfrac{16}{7},$$

and from the second $x = 0$, or $\frac{5}{7}$.

EXAMPLES FOR PRACTICE.

Find the values of the unknown quantity in each of the following examples :—

	Answers.
1. $x^2 - 8x = 9.$	$x = 9$ or $-1.$
2. $x^2 - x = 210.$	$x = 15$ or $-14.$
3. $3x^2 + 2x = 56.$	$x = 4$ or $-\frac{14}{3}.$
4. $\frac{1}{2}x^2 - \frac{1}{3}x = \frac{5}{8}.$	$x = \frac{3}{2}$ or $-\frac{5}{6}.$
5. $\frac{1}{2}x - \frac{1}{3}\sqrt{x} = 22\frac{1}{6}.$	$x = 49$ or $\frac{361}{9}.$
6. $x^6 + 20x^3 = 69.$	$x = \sqrt[3]{3}$ or $\sqrt[3]{-23}.$
7. $2x^4 - x^2 = 3.$	$x = \frac{1}{2}\sqrt{6}$ or $\sqrt{-1}.$
8. $5x^{\frac{1}{4}} - 3x^{\frac{1}{2}} = \frac{4}{3}.$	$x = 3\frac{13}{81}$ or $\frac{1}{81}.$
9. $3x^{2n} - 2x^n = 8.$	$x = (2)^{\frac{1}{n}}$ or $(-\frac{4}{3})^{\frac{1}{n}}.$
10. $-x^2 + x = \frac{6}{25}.$	$x = \frac{2}{5}$ or $\frac{3}{5}.$

	Answers.
11. $x^2 - 4x = -1.$	$x = 2 \pm \sqrt{3}.$
12. $x^{\frac{5}{6}} + x^{\frac{5}{3}} = 1056.$	$x = 64 \text{ or } (-33)^{\frac{6}{5}}.$
13. $x^{\frac{4}{3}} - \frac{1}{2}x^{\frac{8}{3}} = -112.$	$x = 8 \text{ or } (-14)^{\frac{3}{4}}.$
14. $\sqrt{x} + \dfrac{4}{\sqrt{x}} = 5.$	$x = 1 \text{ or } 16.$
15. $\dfrac{48}{x^2} + \dfrac{32}{x} = 11.$	$x = 4 \text{ or } -1\frac{1}{11}.$
16. $x + \sqrt{(5x + 10)} = 8.$	$x = 18 \text{ or } 3.$
17. $\dfrac{1}{x}\sqrt{(1 - x^3)} = x^2.$	$x = \left(\dfrac{-1 \pm \sqrt{5}}{2}\right)^{\frac{1}{3}}.$
18. $x\sqrt{\left(\dfrac{6}{x} - x\right)} = \dfrac{1 + x^2}{\sqrt{x}}.$	$x = \left(\dfrac{2 \pm \sqrt{2}}{2}\right)^{\frac{1}{2}}.$
19. $\sqrt{(10 + x)} - \sqrt[4]{(10 + x)} = 2.$	$x = 6 \text{ or } -9.$
20. $\frac{2}{3}x\sqrt{(3 + 2x^2)} = \frac{1}{2} + \frac{2}{3}x^2.$	$x = \frac{1}{2}\sqrt{(-3 \pm 3\sqrt{2})}.$
21. $x\sqrt{\left(\dfrac{a}{x} - 1\right)} = \sqrt{(x^2 - b^2)}.$	$x = \dfrac{a \pm \sqrt{(a^2 + 8b^2)}}{4}.$
22. $\dfrac{1}{x - 1} - \dfrac{1}{x + 3} = \dfrac{1}{35}.$	$x = 11 \text{ or } -13.$
23. $\dfrac{2x + 9}{9} + \dfrac{4x - 3}{4x + 3} = \dfrac{3x + 38}{18}.$	$x = 6 \text{ or } -\frac{19}{4}.$
24. $x^2 + x + 2\sqrt{(x^2 + x + 4)} = 20.$	$x = 3, -4 \text{ or } \dfrac{-1 \pm \sqrt{129}}{2}.$
25. $\sqrt{(1 + x - x^2)} - 2(1 + x - x^2) = \frac{1}{9}.$	$x = \dfrac{3 \pm \sqrt{41}}{6} \text{ or } \dfrac{3 \pm 2\sqrt{11}}{6}.$
26. $\sqrt{\left(x - \dfrac{1}{x}\right)} + \sqrt{\left(1 - \dfrac{1}{x}\right)} = x.$	$x = \dfrac{1 \pm \sqrt{5}}{2}.$
27. $x^{4n} - 2x^{3n} + x^n = 6.$	$x = \left(\dfrac{1 \pm \sqrt{13}}{2}\right)^{\frac{1}{n}} \text{ or } \left(\dfrac{1 \pm \sqrt{-7}}{2}\right)^{\frac{1}{n}}.$

	Answers.
28. $\{(x-2)^2-x\}^2-(x-2)^2=88-(x-2)$.	$x=6, -1$ or $\frac{5\pm3\sqrt{-3}}{2}$.
29. $x+4+\left(\frac{x+4}{x-4}\right)^{\frac{1}{2}}=\frac{12}{x-4}$.	$x=\pm 5$ or $\pm4\sqrt{2}$.
30. $\left(\frac{x-2}{x+2}\right)^{\frac{1}{2}}+\left(\frac{x+2}{x-2}\right)^{\frac{1}{2}}=4$.	$x=\pm\frac{4}{3}\sqrt{3}$.
31. $x+1=\frac{10}{\sqrt{x}}$.	$x=4$, or $-3\pm4\sqrt{-1}$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To find two numbers such that their difference shall be 8, and their product 240.

Let $x=$ the less number, then $x+8=$ the greater, and $x(x+8)=240$, or $x^2+8x=240$, by the question.

$$\text{Whence } x=-4+\sqrt{(16+240)}=-4+\sqrt{256}$$

$$=-4+16=12=\text{ the less number,}$$

$$\text{and } x+8=12+8=20=\text{ the greater.}$$

2. It is required to divide the number 60 into two such parts, that their product shall be 864.

Let $x=$ one part, then $60-x=$ the other, hence

$$x(60-x)=864, \text{ or } 60x-x^2=864,$$

$$\text{or } x^2-60x=-864.$$

$$\text{Whence } x=30\pm\sqrt{(900-864)}=30\pm\sqrt{36}$$

$$=30\pm6=36 \text{ or } 24;$$

hence 36 and 24 are the parts required.

3. Find two numbers such that their sum shall be $10(a)$, and the sum of their squares $58(b)$.

Let $x=$ the greater number, then $a-x=$ the less, and $x^2+(a-x)^2=2x^2-2ax+a^2=b$, by the question;

$$\therefore x^2 - ax = \frac{b - a^2}{2},$$

$$\text{and } x = \frac{a}{2} + \sqrt{\left(\frac{a^2}{4} + \frac{b - a^2}{2}\right)} = \frac{a}{2} + \tfrac{1}{2}\sqrt{(2b - a^2)}.$$

If $a = 10$, and $b = 58$, then we have

$x = 5 + \frac{1}{2}\sqrt{(116 - 100)} = 5 + 2 = 7$, the greater number,
and $10 - x = 10 - 7 = 3$, the less number.

4. Having sold a piece of cloth for L.24, I gained as much per cent. as it cost me; what was the price of the cloth?

Let $x =$ the price of the cloth in pounds; then

$24 - x =$ the gain on x pounds; therefore

$100 : x :: x : 24 - x$, by the question,

or $x^2 = 2400 - 100x$, or $x^2 + 100x = 2400$;

whence $x = -50 + \sqrt{(2500 + 2400)} = -50 + 70 = 20$;
and the buying price of the cloth was L.20.

5. A person bought a number of sheep for L.80, and if he had bought four more for the same money, he would have paid L.1 less for each: how many did he buy?

Let $x =$ the number bought; then

$\frac{80}{x} =$ price of each, and

$\frac{80}{x+4} =$ price of each, if $x + 4$ cost L.80;

$\therefore \frac{80}{x} - \frac{80}{x+4} = 1$, or $\frac{1}{x} - \frac{1}{x+4} = \frac{1}{80}$, by the question;

$\therefore \frac{4}{x(x+4)} = \frac{1}{80}$, or $x^2 + 4x = 320$;

whence $x = -2 \pm \sqrt{(4 + 320)} = -2 \pm \sqrt{324}$
$= -2 \pm 18 = 16$ or -20.

Therefore the number of sheep bought was 16.

Note.—This example will afford an excellent illustration of the interpretation of the negative value of x. It will be seen that -20 is a root of the equation $x^2 + 4x = 320$, since when substituted for x in it, the equation is verified; but -20 is obviously inadmissible as an answer to the proposed question.

If, however, we inquire to what question the negative value of x, with its sign changed, is an answer, we might proceed by reversing the steps of the preceding solution in the following manner :—

Changing the sign of x in the equation $x^2 + 4x = 320$, we get the equation, $x^2 - 4x = 320$, and the positive root of this equation is 20. But the equation $x^2 - 4x = 320$ is derived from

$$\frac{1}{x-4} - \frac{1}{x} = \frac{1}{80} \text{ or } \frac{80}{x-4} - \frac{80}{x} = 1,$$

and this equation leads us to the following question.

A person bought a number of sheep for L.80, and if he had bought four *fewer* for the same money, he would have paid L.1 *more* for each; how many did he buy?

Thus we see that the negative value of x in the question, taken positively, is the proper answer to the question which has just been enunciated.

6. It is required to find two numbers, such that their sum, product, and difference of their squares, shall be all equal to each other.

Let $x =$ the greater number, and $y =$ the less; then by the question we have

$$x + y = xy \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$x + y = x^2 - y^2 \quad . \quad . \quad . \quad (2).$$

Dividing both members of equation (2) by $x + y$, gives

$$1 = x - y, \text{ or } x = y + 1 \quad . \quad . \quad . \quad (3)$$

Substituting the value of x from (3) in (1) gives

$$2y + 1 = y(y + 1) = y^2 + y, \text{ or } y^2 - y = 1;$$

$$\text{whence } y = \tfrac{1}{2} + \sqrt{(\tfrac{1}{4} + 1)} = \frac{1 + \sqrt{5}}{2},$$

$$\text{and} \quad x = y + 1 \quad = \frac{3 + \sqrt{5}}{2}.$$

7. Find two numbers, such that their sum shall be 13 (a), and the sum of their fourth powers 4721 (b).

Let x = the one number, and y = the other; then

$$x + y = a \quad . \; . \; . \; . \; . \; . \quad (1)$$
$$x^4 + y^4 = b \quad . \; . \; . \; . \; . \; . \quad (2)$$

Raise the first equation to the fourth power, add the second to the result, and divide by 2; then we have .

$$x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4 = \frac{a^4 + b}{2},$$

$$\text{or,} \qquad (x^2 + xy + y^2)^2 = \frac{a^4 + b}{2},$$

$$\therefore \; x^2 + xy + y^2 = \sqrt{\left(\frac{a^4 + b}{2}\right)} = \tfrac{1}{2}\sqrt{2\,(a^4 + b)}.$$

But $x^2 + 2xy + y^2 = a^2$;

hence $xy = a^2 - \frac{1}{2}\sqrt{2(a^4 + b)}$, or $4xy = 4a^2 - 2\sqrt{2(a^4 + b)}$,

$$\therefore \; x^2 - 2xy + y^2 = -3a^2 + 2\sqrt{2(a^4 + b)},$$

$$\text{or } x - y = \{-\,3a^2 + 2\sqrt{2\,(a^4 + b)}\}^{\frac{1}{2}}.$$

But since $x + y = a$, we have by addition and subtraction,

$$x = \frac{a}{2} + \tfrac{1}{2}\{-\,3a^2 + 2\sqrt{2\,(a^4 + b)}\}^{\frac{1}{2}},$$

$$y = \frac{a}{2} - \tfrac{1}{2}\{-\,3a^2 + 2\sqrt{2\,(a^4 + b)}\}^{\frac{1}{2}}.$$

If $a = 13$ and $b = 4721$; then we have $x = 8$ or 5, and $y = 5$ or 8.

Otherwise thus:

Let x = the difference of the numbers; then

$$\frac{a + x}{2} = \text{the greater, and } \frac{a - x}{2} = \text{the less};$$

$$\therefore \; \frac{(a + x)^4}{16} + \frac{(a - x)^4}{16} = b, \text{ or } (a + x)^4 + (a - x)^4 = 16b.$$

Hence $2a^4 + 12a^2x^2 + 2x^4 = 16b$,

$$\therefore \; x^4 + 6a^2x^2 = 8b - a^4;$$

$$\therefore \; x^2 = -\,3a^2 + \sqrt{(9a^4 + 8b - a^4)} = -\,3a^2 + 2\sqrt{2(a^4 + b)};$$

$\therefore \; x = \{-\,3a^2 + 2\sqrt{2(a^4 + b)}\}^{\frac{1}{2}}$, and hence

both $\frac{a + x}{2}$ and $\frac{a - x}{2}$ become known.

EXAMPLES FOR PRACTICE.

1. It is required to divide the number 40 into two such parts, that the sum of their squares shall be 818.

Ans. 23 and 17.

2. To find a number such, that if it be subtracted from 10, and the remainder multiplied by the number itself, the product shall be 21.

Ans. 7 or 3.

3. It is required to divide the number 24 into two such parts, that their product shall be equal to 35 times their difference.

Ans. 10 and 14.

4. It is required to divide the number 20 into two such parts, that twice the square of the greater part shall exceed three times the square of the less, by 96. Ans. 12 and 8.

5. It is required to divide the number 60 into two such parts, that their product shall be to the sum of their squares in the ratio of 2 to 5.

Ans. 20 and 40.

6. It is required to divide the number 146 into two such parts, that the difference of their square roots shall be 6.

Ans. 25 and 121.

7. It is required to find two numbers, such that their sum shall be 23, and their product $116\frac{1}{4}$. Ans. $7\frac{1}{2}$ and $15\frac{1}{2}$.

8. The sum of two numbers is $1\frac{1}{3}$, and the sum of their reciprocals $3\frac{1}{5}$; required the numbers. Ans. $\frac{1}{2}$ and $\frac{5}{6}$.

9. The difference of two numbers is 15, and half their product is equal to the cube of the less number; required the numbers.

Ans. 3 and 18.

10. The difference of two numbers is 5, and the difference of their cubes 1685; required the numbers. Ans. 8 and 13.

11. A person bought a quantity of cloth for L.33, 15s., which he sold at L.2, 8s. per piece, and gained by the bargain as much as one piece cost him; find the number of pieces. Ans. 15.

12. What two numbers are those, whose sum multiplied by the greater is equal to 77, and whose difference multiplied by the less is equal to 12?

Ans. 4 and 7.

13. A grazier bought as many sheep as cost him L.60, and after reserving 15 out of the number, sold the remainder for L.54, and gained 2s. a-head by them; how many sheep did he buy?

Ans. 75.

14. Find two numbers such, that their product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes. Ans. $\frac{1}{2}\sqrt{5}$ and $\frac{1}{4}(5 + \sqrt{5}.)$

15. A company at a tavern had L.8, 15s. to pay for their reckoning; but, before the bill was settled, two of the company left, and those who remained had each 10s. more to pay than before; how many were in the company? Ans. 7.

16. A person ordered L.7, 4s. to be distributed among some poor people, but before the money was divided, two claimants more came in, unexpectedly, by which circumstance the former received each a shilling less than they would otherwise have done; what was the number of claimants at first? Ans. 16.

17. The sum of two numbers is 11, and the sum of their fifth powers is 17831; required the numbers. Ans. 4 and 7.

18. Find two numbers such, that the square of the first together with their product shall be 140, and the square of the second diminished by their product, 78. Ans. 7 and 13.

19. Two detachments of foot being ordered to a station at the distance of 39 miles from their present quarters, began their march at the same time; but one party, by travelling $\frac{1}{4}$ of a mile an hour more than the other, arrived there an hour sooner; find their rates of marching.

Ans. $3\frac{1}{4}$ and 3 miles per hour.

20. The continued product of four consecutive numbers is 3024; find the numbers. Ans. 6, 7, 8, 9.

21. A and B start at the same time to travel 150 miles, A travels 3 miles an hour more than B, and finishes his journey 8 hours 20 minutes before him; what are their rates of travelling?

Ans. 9 and 6 miles per hour.

22. A vintner sold 7 dozen of sherry and 12 dozen of claret for L.50; and he sold 3 dozen more of sherry for L.10, than he did of claret for L.6; find the price of each.

Ans. sherry L.2, and claret L.3 per dozens.

23. If r_1, r_2 be the roots of the quadratic equation $x^2 + ax + b = 0$, prove that

$$r_1 + r_2 = -a, \text{ and } r_1 r_2 = b.$$

24. Two mail trains start at the same time, the one from London to Edinburgh, and the other from Edinburgh to London; now, if the London train arrives in Edinburgh a hours, and the Edinburgh train in London b hours, after they passed each other; show that the respective times of completing the whole distance by the London and Edinburgh trains are

$$a^{\frac{1}{2}}(a^{\frac{1}{2}} + b^{\frac{1}{2}}) \text{ and } b^{\frac{1}{2}}(a^{\frac{1}{2}} + b^{\frac{1}{2}}).$$

25. The product of a number consisting of two digits by the digit on the left hand is 192, and if the number be reversed, and multiplied by the digit now on the left, the product is 672; required the number.

Ans. 48.

26. Two partners, A and B, gained L.18, 15s. by trade; A's money was in trade 12 months, and he received for his principal and gain L.26; and B's money, which was L.30, was in trade 17 months; find A's stock.

Ans. L.20.

27. The difference of two numbers is 8, and the sum of their fourth powers is 14722; required the numbers. Ans. 3 and 11.

SIMULTANEOUS EQUATIONS OF THE SECOND AND HIGHER DEGREES,

Containing two or more unknown quantities.

1. Given $x + y = a$, and $xy = b$, to find the values of x and y.

Squaring the first equation, we have

$$\begin{array}{rrcl} & x^2 + 2xy + y^2 & = & a^2, \\ \text{and by the second,} & 4xy & = & 4b \\ \hline \therefore & x^2 - 2xy + y^2 & = & a^2 - 4b, \\ \text{or} & x - y & = & \sqrt{(a^2 - 4b)}, \\ \text{but} & x + y & = & a, \end{array}$$

$$\text{hence } 2x = a + \sqrt{(a^2 - 4b)}, \text{ or } x = \frac{a + \sqrt{(a^2 - 4b)}}{2};$$

$$2y = a - \sqrt{(a^2 - 4b)}, \text{ or } y = \frac{a - \sqrt{(a^2 - 4b)}}{2}.$$

2. Given $x^2 + y^2 = a^2$, and $x + y = b$, to find the values of x and y.

Squaring the second equation, gives

$$\begin{aligned} x^2 + 2xy + y^2 &= b^2 \dots (1) \\ \text{but } 2x^2 \qquad + 2y^2 &= 2a^2 \dots (2) \end{aligned}$$

Subtracting equation (1) from (2), and taking the square root, gives

$$x - y = \sqrt{(2a^2 - b^2)},$$

$$\text{but } x + y = b,$$

$$\therefore\ 2x = b + \sqrt{(2a^2 - b^2)}, \text{ or } x = \frac{b + \sqrt{(2a^2 - b^2)}}{2},$$

$$2y = b - \sqrt{(2a^2 - b^2)}, \text{ or } y = \frac{b - \sqrt{(2a^2 - b^2)}}{2}.$$

3. Given $x^3 + y^3 = a^3$, and $x + y = b$, to find the values of x and y.

By dividing the first of these equations by the second, we have

$$x^2 - xy + y^2 = \frac{a^3}{b},$$

and squaring the second, $x^2 + 2xy + y^2 = b^2$;

$$3xy = b^2 - \frac{a^3}{b} = \frac{b^3 - a^3}{b}, \text{ or } xy = \frac{b^3 - a^3}{3b}.$$

Whence, by a process similar to that employed in the first example, we have

$$x = \tfrac{1}{2}\left\{ b + \left(\frac{4a^3 - b^3}{3b}\right)^{\frac{1}{2}} \right\} \text{ and } y = \tfrac{1}{2}\left\{ b - \left(\frac{4a^3 - b^3}{3b}\right)^{\frac{1}{2}} \right\}.$$

4. Given $x^4 + y^4 = 17$, and $x + y = 3$, to find the values of x and y.

Raising the second equation to the fourth power, we get

$$\begin{aligned} x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 &= 81; \\ \text{but, } x^4 \qquad\qquad\qquad\qquad + y^4 &= 17, \end{aligned}$$

Adding and dividing by 2, $x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4 = 49$.

Taking the square root, $x^2 + xy + y^2 = 7$.

Squaring the second equation, $x^2 + 2xy + y^2 = 9$,

$$\therefore\ xy = 2,$$

whence, by the first example, we have $x = 2$, and $y = 1$.

5. Given $x^4 + x^2y^2 + y^4 = 91$, and $x^2 + xy + y^2 = 13$, to find the values of x and y.

Dividing the first equation by the second, gives

$$x^2 - xy + y^2 = 7,$$
$$\text{but } x^2 + xy + y^2 = 13.$$

Adding and subtracting, we have

$$x^2 + y^2 = 10 \text{ and } 2xy = 6.$$

Adding and subtracting, and taking the square root, give

$$x + y = 4 \text{ and } x - y = 2.$$

And again adding and subtracting,

$$x = 3 \text{ and } y = 1.$$

6. Given $x^2 + 3xy + 4y^2 = 14$, and $3x^2 + 4xy + 5y^2 = 25$, to find x and y.

Let $x = vy$; then by substitution these equations become

$$y^2(v^2 + 3v + 4) = 14 \ldots\ldots (1)$$
$$y^2(3v^2 + 4v + 5) = 25 \ldots\ldots (2)$$

Dividing (1) by (2), and simplifying the result, we get

$$v^2 - \tfrac{19}{17}v = \tfrac{30}{17}; \text{ whence } v = 2 \text{ or } -\tfrac{15}{17}.$$

Taking the value $v = 2$, we get from (1)

$$y^2 = 1, \text{ or } y = 1, \text{ and thence } x = vy = 2.$$

7. Given $x^3 + y^3 + x^2y + xy^2 = 65$, and $x^4y^2 + x^2y^4 = 468$, to find x and y.

These equations may be written thus:

$$(x + y)(x^2 + y^2) = 65.$$
$$x^2y^2 \ (x^2 + y^2) = 468.$$

Let now $x = vy$; then they become

$$y^3(v + 1)(v^2 + 1) = 65 \ldots\ldots (1)$$
$$v^2y^6(v^2 + 1) = 468 \ldots\ldots (2)$$

Squaring equation (1) and dividing by (2) gives

$$\frac{(v+1)^2(v^2+1)}{v^2} = \frac{4225}{468} = \frac{325}{36},$$

$$\text{or } (v + v^{-1})^2 + 2(v + v^{-1}) = \frac{325}{36}.$$

Solving this quadratic, we have

$$v + v^{-\frac{1}{6}} + 1 = \pm 3\tfrac{1}{6}, \text{ or } v + \frac{1}{v} = 2\tfrac{1}{6} \text{ or } -4\tfrac{1}{6}.$$

Solving the quadratic $v + \dfrac{1}{v} = 2\tfrac{1}{6}$, gives, $v = \tfrac{3}{2}$ or $\tfrac{2}{3}$, and hence by (1),

$$y^3 = 8 \text{ or } 27, \ \therefore y = 2 \text{ or } 3, \text{ and } x = vy = 3 \text{ or } 2.$$

EXAMPLES FOR PRACTICE.

	Answers.
1. $x - y = 5$, and $xy = 150$.	$x = 15,\ y = 10$.
2. $x^2 + y^2 = 113$, and $xy = 56$.	$x = \pm 8,\ y = \pm 7$.
3. $xy + y = 10$, and $x^2y^2 + y^2 = 68$.	$x = 4,\ y = 2$.
4. $x^2 + xy = 84$, and $x^2 - y^2 = 24$.	$x = \pm 7,\ y = \pm 5$.
5. $x^2y^2 + 8xy = 180$, and $x + 3y = 11$.	$x = 5$ or 6, $y = 2$ or $\frac{5}{3}$.
6. $\dfrac{2}{x} + \dfrac{3}{y} = 8$, and $7xy = 6$.	$x = 2$ or $\frac{2}{7}$, $y = \frac{3}{7}$ or 3.
7. $2x^2 + 3xy = 26$, and $3y^2 + 2xy = 39$.	$x = \pm 2,\ y = \pm 3$.
8. $x + y + \sqrt{(x+y)} = 12$, and $x^2 + y^2 = 41$.	$x = 5,\ y = 4$.
9. $\left.\begin{aligned} \frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x} &= \frac{27}{4} \\ x - y &= 2 \end{aligned}\right\}$	$x = 4, -2$, or $1 \pm \sqrt{\frac{3}{11}}$ $y = 2, -4$, or $-1 \pm \sqrt{\frac{3}{11}}$.
10. $\left.\begin{aligned} x^2 + xy + y^2 &= 91 \\ x + \sqrt{xy} + y &= 13 \end{aligned}\right\}$	$x = 9$ or 1 $y = 1$ or 9.

	Answers.
11. $\left.\begin{aligned} xy &= 5(x+y) \\ x^2y^2 &= 49(x^2+y^2) \end{aligned}\right\}$	$x = \frac{35}{3}$ or $\frac{35}{4}$ $y = \frac{35}{4}$ or $\frac{35}{3}$.
12. $\left.\begin{aligned} x^2 + x\sqrt[3]{xy^2} &= 208 \\ y^2 + y\sqrt[3]{x^2y} &= 1053 \end{aligned}\right\}$	$x = \pm\ 8$ $y = \pm\ 27$.
13. $\left.\begin{aligned} \left(\frac{3x}{x+y}\right)^{\frac{1}{2}} + \left(\frac{x+y}{3x}\right)^{\frac{1}{2}} &= 2 \\ xy - (x+y) &= 54 \end{aligned}\right\}$	$x = 6$ or $-\frac{9}{2}$ $y = 12$ or -9.
14. $\left.\begin{aligned} x + y + z &= 14 \\ x^2 + y^2 + z^2 &= 84 \\ xz &= y^2 \end{aligned}\right\}$	$x = 8$ or 2 $y = 4$ $z = 2$ or 8.

THE SQUARE ROOT OF A BINOMIAL SURD.

The method of extracting the square root of a binomial surd, one of whose terms is rational, and the other a quadratic surd, has already been given at page 72, and as we are now in possession of principles to enable us to investigate the method, we propose to give it here. The investigation depends on the following propositions :—

(1) *The product of two dissimilar quadratic surds will also be a quadratic surd.*

For, if possible, let $\sqrt{x} \times \sqrt{y} = h$, a rational quantity ; then $xy = h^2$;

therefore $y = \dfrac{h^2}{x} = \dfrac{h^2}{x^2} \cdot x$, and $\sqrt{y} = \dfrac{h}{x}\sqrt{x}$;

and consequently $\sqrt{x}$ and $\sqrt{y}$ are *similar* quadratic surds, which is contrary to the hypothesis.

(2) *A surd cannot be equal to the sum or difference of a rational quantity and a surd, or to the sum or difference of two dissimilar surds.*

For let $\sqrt{x} = a \pm \sqrt{y}$; then $x = a^2 \pm 2a\sqrt{y} + y$, and hence

$$\pm \sqrt{y} = \frac{x - y - a^2}{2a},$$

therefore a surd is equal to a rational quantity, which is absurd.

Again, let $\sqrt{x} = \sqrt{a} \pm \sqrt{y}$; then $x = a \pm 2\sqrt{ay} + y$; and hence

$$\pm\sqrt{ay} = \frac{x - y - a}{2};$$

therefore the product of two dissimilar surds is equal to a rational quantity, which is absurd, by (1).

(3) *In any equation $a + \sqrt{b} = x + \sqrt{y}$, consisting of rational quantities and surds, the rational quantities on each side are equal, and likewise the irrational quantities.*

For by transposition we have

$$\sqrt{b} = (x - a) + \sqrt{y};$$

hence, if x be not equal to a, or $x - a$ not $= 0$, we shall have a surd equal to the sum of a rational quantity and a surd, which is impossible, by (2). Consequently if $a + \sqrt{b} = x + \sqrt{y}$; then

$$x = a \text{ and } \sqrt{y} = \sqrt{b}.$$

From these principles we can deduce the formula for extracting the square root of a binomial surd of the form $a \pm \sqrt{b}$.

Let $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$; then $a + \sqrt{b} = x + y + 2\sqrt{xy}$; hence by (3)

$$x + y = a \text{ and } 2\sqrt{xy} = \sqrt{b};$$

$$\therefore \quad x^2 + 2xy + y^2 = a^2,$$

$$4xy \qquad = b,$$

Subtracting, gives $x^2 - 2xy + y^2 = a^2 - b$,

$$\text{or} \quad x - y = \sqrt{(a^2 - b)};$$

$$\text{but} \quad x + y = a;$$

$$\therefore 2x = a + \sqrt{(a^2 - b)} \text{ or } x = \frac{a + \sqrt{(a^2 - b)}}{2},$$

$$2y = a - \sqrt{(a^2 - b)} \text{ or } y = \frac{a - \sqrt{(a^2 - b)}}{2};$$

hence $\sqrt{a+\sqrt{b}} = \sqrt{x}+\sqrt{y} = \sqrt{\dfrac{a+\sqrt{(a^2-b)}}{2}} + \sqrt{\dfrac{a-\sqrt{(a^2-b)}}{2}}$.

Similarly $\sqrt{a-\sqrt{b}} = \sqrt{x}-\sqrt{y} = \sqrt{\dfrac{a+\sqrt{(a^2-b)}}{2}} - \sqrt{\dfrac{a-\sqrt{(a^2-b)}}{2}}$.

We shall only give an example or two in illustration here, and refer the student to those at p. 73 and p. 78, Ex. 6.

EXAMPLES.

1. Find the square root of $2n\sqrt{-1}$.

Let $\sqrt{(2n\sqrt{-1})} = \sqrt{x} + \sqrt{y}$; then, by squaring,

$$2n\sqrt{-1} = x + y + 2\sqrt{xy};$$

$$\therefore \quad x + y = 0, \text{ and } 2\sqrt{xy} = 2n\sqrt{-1};$$

$$\text{hence} \quad x^2 + 2xy + y^2 = 0,$$

$$4xy = -4n^2;$$

$$\therefore \quad x^2 - 2xy + y^2 = 4n^2, \text{ and } x - y = 2n.$$

From the two equations $x + y = 0$, and $x - y = 2n$, we get

$$x = n, \text{ and } y = -n, \therefore \sqrt{(2n\sqrt{-1})} = \sqrt{n} + \sqrt{-n}.$$

2. Find the square root of $4\frac{1}{3} - \frac{4}{3}\sqrt{3}$ and of $-2\sqrt{-1}$.

Ans. $2 - \frac{1}{3}\sqrt{3}$ and $1 - \sqrt{-1}$.

3. Find the square root of $2\frac{1}{4} - \sqrt{5}$ and the fourth root of $49 + 20\sqrt{6}$.

Ans. $1 - \frac{1}{2}\sqrt{5}$ and $\sqrt{2} + \sqrt{3}$.

4. Find the square root of $4mn + 2(m^2 - n^2)\sqrt{-1}$, and of $1 + \sqrt{1-a^2}$.

Ans. $m + n + (m - n)\sqrt{-1}$, and $\sqrt{\left(\dfrac{1+a}{2}\right)} + \sqrt{\left(\dfrac{1-a}{2}\right)}$.

CUBIC EQUATIONS.

A *cubic equation* is one in which the highest power of the unknown quantity is the third power, and, like a quadratic, is either simple or compound.

A simple cubic equation is of the form $ax^3 = b$, which solved gives

$$x^3 = \frac{b}{a}, \text{ and } x = \left(\frac{b}{a}\right)^{\frac{1}{3}}.$$

A compound cubic equation is of either of the forms

$$x^3 + ax = b, \; x^3 + ax^2 = b, \text{ or } x^3 + ax^2 + bx = c.$$

The two latter forms may easily be reduced to the first by the following

Rule.—Take a new unknown quantity, and increase or diminish it by a third part of the coefficient of the second term of the equation with its sign changed; then if this sum or difference be substituted for the original unknown quantity and its powers in the proposed equation, there will result an equation of the first of the above forms wanting the second power of the unknown quantity.

Note.—The second term of a complete equation of any degree may be removed in a similar manner, by substituting for the unknown quantity the sum or difference, as above, of a new unknown quantity, and the 2d, 3d, 4th, 5th, *etc.*, part of the coefficient of its second term with its sign changed, according as the equation is of the 2d, 3d, 4th, 5th, *etc.*, degree.

EXAMPLES.

1. Transform the equation $x^3 + 3ax^2 - b = 0$ into another, that shall want the second power of the unknown quantity.

Let $x = y - \frac{3a}{3} = y - a$; then by substitution

$$\begin{array}{rrrrr}
x^3 = & y^3 & -\ 3ay^2 & +\ 3a^2y & -\ a^3 \\
3ax^2 = & & 3ay^2 & -\ 6a^2y & +\ 3a^3 \\
-b = & & & & -\ b \\
\hline
\text{By addition,} & y^3 & & -\ 3a^2y & +\ 2a^3 - b = 0;
\end{array}$$

hence the transformed equation is $y^3 - 3a^2y = b - 2a^3$, in which the second power of y is wanting.

This transformation may be effected in a more convenient manner than by substitution. For if the given equation be divided by $x + a$, we shall have for quotient $x^2 + 2ax - 2a^2$ and for remainder $2a^3 - b$; hence

$$x^3 + 3ax^2 - b = (x^2 + 2ax - 2a^2)(x + a) + 2a^3 - b \quad . \quad . \quad (1)$$

Again, dividing $x^2 + 2ax - 2a^2$ by $x + a$, we get for quotient $x + a$ and for remainder $-3a^2$; hence $x^2 + 2ax - 2a^2 = (x + a)(x + a) - 3a^2 = (x + a)^2 - 3a^2$, and equation (1) becomes

$$\begin{aligned}
x^3 + 3ax^2 - b &= \{(x + a)^2 - 3a^2\}(x + a) + 2a^3 - b \\
&= (x + a)^3 - 3a^2\,(x + a) + 2a^3 - b = 0.
\end{aligned}$$

Now if $x + a$ be denoted by y; then the last equation will become

$$y^3 - 3a^2y = b - 2a^3,$$

in which the second power of y is wanting, as in the preceding method.

Uniting these operations, and using detached coefficients and the synthetic mode of division, the whole work will be arranged as follows:

$$\begin{array}{llll}
1 & +\ 3a & +\ 0 & -\ b\ (-\ a \\
 & -\ a & -\ 2a^2 & +\ 2a^3 \\
\hline
1 & +\ 2a & -\ 2a^2 & +\ (2a^3 - b) \\
 & -\ a & -\ a^2 & \\
\cline{1-3}
1 & +\ a & -\ 3a^2 & \\
 & -\ a & & \\
\cline{1-2}
1 & +\ 0 & &
\end{array}$$

Hence $y^3 - 3a^2y + 2a^3 - b = 0$, or $y^3 - 3a^2y = b - 2a^3$, as before.

2. Let the equation $x^3 - 12x^2 + 3x + 16 = 0$ be transformed into another that shall want the second power of the unknown quantity.

Let $x = y + \frac{12}{3} = y + 4$; then $y = x - 4$, and dividing successively by $x - 4$, we have the following process:

$$\begin{array}{rrrrl}
1 & -\;12 & +\;\;3 & +\;\;16 & (\,+\,4 \\
 & +\;\;4 & -\;32 & -\;116 & \\
\hline
1 & -\;\;8 & -\;29 & -\;100 & \\
 & +\;\;4 & -\;16 & & \\
\hline
1 & -\;\;4 & -\;45 & & \\
 & +\;\;4 & & & \\
\hline
1 & +\;\;0 & & &
\end{array}$$

$\therefore y^3 - 45\,y = 100$ is the equation required.

3. Transform the following equations into others that shall want the second power of the unknown quantity.

(1.) $x^3 - 6x^2 = 10$. Ans. $y^3 - 12y = 26$.

(2.) $y^3 - 15y^2 + 81y = 243$. $x^3 + 6x = 88$.

(3.) $x^3 + \frac{3}{4}x^2 + \frac{7}{8}x = \frac{9}{16}$. $y^3 + \frac{11}{16}y = \frac{3}{4}$.

(4.) $2x^3 - 3x^2 + 4x = 5$. $y^3 + \frac{5}{4}y = \frac{7}{4}$.

SOLUTION OF CUBIC EQUATIONS.

Let the given cubic equation be $x^3 + ax = b$, a form to which both the other forms can be reduced, and let $x = y + z$; then by substitution in the given equation, we shall have

$$y^3 + 3yz\,(y + z) + z^3 + a\,(y + z) = b.$$

Now, if $3yz = -a$; then we get $y^3 + z^3 = b$, and by substituting in this the value of z from the former, viz. $z = -\frac{\frac{1}{3}a}{y}$, we get readily $y^6 - by^3 = \frac{1}{27}a^3$; and solving this quadratic gives

$$y = \sqrt[3]{\left\{\frac{b}{2} + \sqrt{\left(\frac{b^2}{4} + \frac{a^3}{27}\right)}\right\}}.$$

Hence $z = -\frac{\frac{1}{3}a}{y}$ is known, and consequently $y + z$, or

$$x = \sqrt[3]{\left\{\frac{b}{2} + \sqrt{\left(\frac{b^2}{4} + \frac{a^3}{27}\right)}\right\}} - \frac{\frac{1}{3}a}{\sqrt[3]{\left\{\frac{b}{2} + \sqrt{\left(\frac{b^2}{4} + \frac{a^3}{27}\right)}\right\}}},$$

$$\text{or } x = \sqrt[3]{\left\{\frac{b}{2} + \sqrt{\left(\frac{b^2}{4} + \frac{a^3}{27}\right)}\right\}} + \sqrt[3]{\left\{\frac{b}{2} - \sqrt{\left(\frac{b^2}{4} + \frac{a^3}{27}\right)}\right\}}.^*$$

The last expression is obtained from the former by multiplying both numerator and denominator of the fractional part by $\sqrt[3]{\left\{\frac{b}{2} - \sqrt{\left(\frac{b^2}{4} + \frac{a^3}{27}\right)}\right\}}$.

When the coefficient a of the first power of x in the given equation is negative, then $\frac{a}{3}$ and its cube $\frac{a^3}{27}$ will be negative; but if the absolute term b be negative, $\frac{b}{2}$ will be negative, and $\frac{b^2}{4}$ will be positive.

When the equation is of the form $x^3 - ax = \pm b$, and $\frac{a^3}{27}$ is greater than $\frac{b^2}{4}$, or $4a^3$ greater than $27b^2$, the solution of it cannot be obtained by the above formula, as the equation in this instance falls under what is usually called the *Irreducible Case*[†] of cubic equations.

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[*] This method of solving cubic equations is usually ascribed to **Cardan**, a celebrated Italian analyst of the 16th century; but the authors of it were **Scipio Ferreus** and **Nicholas Tartalea**, who discovered it about the same time independently of each other, as is proved by **Montucla**, in his *Histoire des Mathématiques*, vol. i. p. 568, and more at large in **Hutton's** *Mathematical Dictionary*, Art. Algebra.

[†] It may here be observed, as a remarkable circumstance in the history of this science, that the solution of the *Irreducible Case* above mentioned, has baffled the united efforts of the most celebrated mathematicians in Europe; although it is well known that all the three roots of the equation are, in this case, real; whereas, in those that are resolvable by the above formula, only one of the roots is real; so that, in fact, the rule is only applicable to such cubics as have two impossible roots, or two equal roots.

When $\frac{a^3}{27}$ is negative, and *equal* to $\frac{b^2}{4}$; then the equation is resolvable by this method; for $\sqrt{\left(\frac{b^2}{4} - \frac{a^3}{27}\right)} = 0$, and

$$x = \sqrt[3]{\frac{b}{2}} + \sqrt[3]{\frac{b}{2}} = \sqrt[3]{(4b)}.$$

In this case, it can easily be shewn that the equation has two equal roots.

EXAMPLES.

1. Given $2x^3 - 12x^2 + 36x = 44$, or $x^3 - 6x^2 + 18x = 22$, to find the value of x.

Remove the second power of the unknown quantity, by dividing successively by $x - \frac{6}{3}$ or $x - 2$, as directed in the previous rule, thus:

$$\begin{array}{rrrrl}
1 & -\,6 & +\,18 & -\,22 & (2 \\
 & +\,2 & -\,8 & +\,20 & \\
\hline
1 & -\,4 & +\,10 & -\,2 & \\
 & +\,2 & -\,4 & & \\
\hline
1 & -\,2 & +\,6 & & \\
 & +\,2 & & & \\
\hline
1 & +\,0 & & &
\end{array}$$

Hence $y^3 + 6y - 2 = 0$, or $y^3 + 6y = 2$ is the transformed equation, in which $y = x - 2$, and therefore $x = y + 2$.

Comparing the equation $y^3 + 6y = 2$ with the equation $y^3 + ay = b$, we have $a = 6$ and $b = 2$; hence by the formula

$$y = \sqrt[3]{\left\{\frac{b}{2} + \sqrt{\left(\frac{b^2}{4} + \frac{a^3}{27}\right)}\right\}} + \sqrt[3]{\left\{\frac{b}{2} - \sqrt{\left(\frac{b^2}{4} + \frac{a^3}{27}\right)}\right\}}$$

$$= \sqrt[3]{\{1 + \sqrt{(1+8)}\}} + \sqrt[3]{\{1 - \sqrt{(1+8)}\}} = \sqrt[3]{4} + \sqrt[3]{-2} = \sqrt[3]{4} - \sqrt[3]{2};$$

hence $x = y + 2 = 2 + \sqrt[3]{4} - \sqrt[3]{2}$,

or $x = 2 + 1{\cdot}587401 - 1{\cdot}259921 = 2{\cdot}32748$.

2. Given $x^3 - 6x = 12$, to find the value of x.

Here $a = -6$ and $b = 12$; therefore by the formula

$$\begin{aligned}
x &= \sqrt[3]{\{6 + \sqrt{(36 - 8)}\}} &&+ \frac{2}{\sqrt[3]{\{6 + \sqrt{(36 - 8)}\}}} \\
&= \sqrt[3]{(6 + 2\sqrt{7})} &&+ \frac{2}{\sqrt[3]{(6 + 2\sqrt{7})}} \\
&= \sqrt[3]{(6 + 5{\cdot}2915026)} &&+ \frac{2}{\sqrt[3]{(6 + 5{\cdot}2915026)}} \\
&= \sqrt[3]{(11{\cdot}2915026)} &&+ \frac{2}{\sqrt[3]{(11{\cdot}2915026)}} \\
&= 2{\cdot}24345 &&+ \frac{2}{2{\cdot}24345} = 2{\cdot}24345 + {\cdot}89148 \\
&= 3{\cdot}13493.
\end{aligned}$$

3. Given $x^3 - 2x = -4$, to find the value of x.

Here $a = -2$ and $b = -4$; therefore by the formula

$$\begin{aligned}
x &= \sqrt[3]{\{-2 + \sqrt{(4 - \tfrac{8}{27})}\}} + \sqrt[3]{\{-2 - \sqrt{(4 - \tfrac{8}{27})}\}} \\
&= \sqrt[3]{(-2 + \tfrac{10}{9}\sqrt{3})} - \sqrt[3]{(2 + \tfrac{10}{9}\sqrt{3})} \\
&= \sqrt[3]{(-2 + 1{\cdot}924501)} - \sqrt[3]{(2 + 1{\cdot}924501)} \\
&= \sqrt[3]{(-{\cdot}075499)} - \sqrt[3]{(3{\cdot}924501)} \\
&= -{\cdot}42265 - 1{\cdot}57734 = -1{\cdot}99999\cdots\cdots = -2.
\end{aligned}$$

In cases of this kind, the cube root of the binomial surd can be extracted in a binomial form: thus $\sqrt[3]{(-2 + \frac{10}{9}\sqrt{3})} = -1 + \frac{1}{3}\sqrt{3}$, and $\sqrt[3]{(2 + \frac{10}{9}\sqrt{3})} = 1 + \frac{1}{3}\sqrt{3}$;

therefore $x = -1 + \frac{1}{3}\sqrt{3} - (1 + \frac{1}{3}\sqrt{3}) = -2$.

Note.—When one of the roots of a cubic equation has been found, either by trial or in any other way, the other two roots may be determined as follows:

Let the known root be denoted by r, and divide the equation $x^3 + ax - b = 0$ by $x \mp r$ according as r is positive or negative; then the quotient will give a quadratic equation whose roots are the other roots of the given cubic equation.

Thus if the equation be $x^3 - 15x = 4$, we can readily find, by a few trials, that 4 is a root; hence $x = 4$ or $x - 4 = 0$; then

$$\begin{array}{l} 1 + 0 - 15 - 4\,(\,4 \\ \quad\;\; 4 + 16 + 4 \\ \hline 1 + 4 + \;\; 1 \quad * \end{array}$$

Hence $x^2 + 4x + 1 = 0$ is the resulting quadratic, and its roots are $-2 + \sqrt{3}$ and $-2 - \sqrt{3}$; therefore the three roots of the given equation are

$$4,\ -2 + \sqrt{3}, \text{ and } -2 - \sqrt{3}.$$

EXAMPLES FOR PRACTICE.

	Answers.
1. $x^3 + 3x^2 - 6x = 8$.	$x = -1, 2$ or -4.
2. $x^3 + x^2 = 500$.	$x = 7{\cdot}6172792$.
3. $x^3 + 12x = 20$.	$x = 1{\cdot}4253514$.
4. $x^3 - 6x = 6$.	$x = \sqrt[3]{2} + \sqrt[3]{4}$.
5. $x^3 + 9x = 6$.	$x = \sqrt[3]{9} - \sqrt[3]{3}$.
6. $x^3 - 22x = 24$.	$x = \sqrt[3]{(12 + \frac{26}{3}\sqrt{-10})} + \sqrt[3]{(12 - \frac{26}{3}\sqrt{-10})}$.
7. $x^3 - 17x^2 + 54x = 350$.	$x = 14{\cdot}9540686$.

BIQUADRATIC EQUATIONS.

A *biquadratic equation* is one in which the highest power of the unknown quantity is the fourth power. The general form is

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

or when the second term is removed by transformation, the form is

$$x^4 + ax^2 + bx + c = 0.$$

This equation may be resolved by either of the following methods:

First method.

Let $x^4 + ax^2 + bx + c = (x^2 + p)^2 - (hx + k)^2$

$$= x^4 + (2p - h^2)x^2 - 2hkx + p^2 - k^2;$$

and that these expressions may be identical, we must have

$$\begin{aligned} 2p - h^2 &= a, \text{ or } h^2 = 2p - a \quad . \quad . \quad . \quad . \quad (1) \\ -2hk &= b, \text{ or } 2hk = -b \quad . \quad . \quad . \quad . \quad . \quad (2) \\ p^2 - k^2 &= c, \text{ or } k^2 = p^2 - c \quad . \quad . \quad . \quad . \quad (3). \end{aligned}$$

Now since four times the product of equations (1) and (3) is equal to the square of (2), we get

$$4(2p - a)(p^2 - c) = b^2,$$

$$\text{or } p^3 - \tfrac{1}{2}ap^2 - cp = \frac{b^2 - 4ac}{8}.$$

Remove the second term from this equation by dividing by $p - \frac{\frac{1}{2}a}{3}$ or $p - \frac{1}{6}a$; thus,

$$\begin{array}{llll} 1 & -\tfrac{1}{2}a & -c & -(\tfrac{1}{8}b^2 - \tfrac{1}{2}ac)\ \big(\tfrac{1}{6}a \\ & +\tfrac{1}{6}a & -\tfrac{1}{18}a^2 & -(\tfrac{1}{108}a^3 + \tfrac{1}{6}ac) \\ \hline 1 & -\tfrac{1}{3}a & -(\tfrac{1}{18}a^2 + c) & -(\tfrac{1}{108}a^3 + \tfrac{1}{8}b^2 - \tfrac{1}{3}ac) \\ & +\tfrac{1}{6}a & -\tfrac{1}{36}a^2 & \\ \cline{1-3} 1 & -\tfrac{1}{6}a & -(\tfrac{1}{12}a^2 + c) & \\ & +\tfrac{1}{6}a & & \\ \cline{1-2} 1 & +0 & & \end{array}$$

Hence $z^3 - (\frac{1}{12}a^2 + c)z = \frac{1}{108}a^3 + \frac{1}{8}b^2 - \frac{1}{3}ac$, is the transformed equation, in which $z = p - \frac{1}{6}a$ or $p = z + \frac{1}{6}a$. Let the root of this cubic in z be denoted by r; then $p = r + \frac{1}{6}a$, and hence from (1) and (3)

$$h = \sqrt{(2p - a)} = \sqrt{\{2(r - \tfrac{1}{3}a)\}},$$

$$k = \sqrt{(p^2 - c)} = -\sqrt{\{(r + \tfrac{1}{6}a)^2 - c\}};$$

where k is *negative*, because by (2) h and k must have contrary signs.

But by hypothesis $(x^2 + p)^2 - (hx + k)^2 = 0$, and therefore

$$x^2 + p = \pm (hx + k),$$

which furnishes the two quadratic equations

$$x^2 - hx = -(p - k), \text{ and } x^2 + hx = -(p + k).$$

Solving these quadratics, and substituting the preceding values of p, h and k, we get

$$\begin{aligned}
x &= \tfrac{1}{2}\sqrt{\{2(r - \tfrac{1}{3}a)\}} + \sqrt{\{-\tfrac{1}{2}r - \tfrac{1}{3}a - \sqrt{[(r + \tfrac{1}{6}a)^2 - c]}\}} \\
x &= \tfrac{1}{2}\sqrt{\{2(r - \tfrac{1}{3}a)\}} - \sqrt{\{-\tfrac{1}{2}r - \tfrac{1}{3}a - \sqrt{[(r + \tfrac{1}{6}a)^2 - c]}\}} \\
x &= -\tfrac{1}{2}\sqrt{\{2(r - \tfrac{1}{3}a)\}} + \sqrt{\{-\tfrac{1}{2}r - \tfrac{1}{3}a + \sqrt{[(r + \tfrac{1}{6}a)^2 - c]}\}} \\
x &= -\tfrac{1}{2}\sqrt{\{2(r - \tfrac{1}{3}a)\}} - \sqrt{\{-\tfrac{1}{2}r - \tfrac{1}{3}a + \sqrt{[(r + \tfrac{1}{6}a)^2 - c]}\}}.^*
\end{aligned}$$

Second method.

Let $x^4 + ax^2 + bx + c = (x^2 + px + q)(x^2 - px + s)$

$$= x^4 + (s + q - p^2)x^2 + p(s - q)x + sq,$$

and that these two expressions may be identical, we must have

$$s + q - p^2 = a, \text{ or } ps + pq = ap + p^3 \quad . \ . \ . \quad (1)$$

$$p(s - q) = b, \text{ or } ps - pq = b \quad . \ . \ . \quad (2)$$

$$sq = c \quad . \ . \ . \quad (3)$$

From the equations (1) and (2) we derive

$$2ps = ap + p^3 + b, \text{ and } 2pq = ap + p^3 - b;$$

$$\therefore\ 4p^2sq = (ap + p^3 + b)(ap + p^3 - b) = 4cp^2, \text{ by (3)}.$$

Multiplying out, and transposing, will give

$$p^6 + 2ap^4 + (a^2 - 4c)p^2 = b^2.$$

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[*] The method of solving biquadratic equations was first discovered by **Louis Ferrari**, a disciple of the celebrated **Cardan**, before mentioned; but the rule here given is derived from that of **Descartes**, published in his Algebra, 1637, and similar to the method of **Thomas Simpson**, given in his Algebra, third edition, page 148, 150, &c. There are other solutions by Waring, Euler, &c on analogous principles.

If $p^2 = z$, then $z^3 + 2az^2 + (a^2 - 4c)z = b^2$, a cubic equation, which will give the value of z, and thence that of p. We have then

$$s = \tfrac{1}{2}\left(a + p^2 + \frac{b}{p}\right) \text{ and } q = \tfrac{1}{2}\left(a + p^2 - \frac{b}{p}\right),$$

which become known since p has been determined. But by hypothesis $(x^2 + px + q)(x^2 - px + s) = 0$; therefore we have either

$$x^2 + px + q = 0, \text{ or } x^2 - px + s = 0.$$

The solution of these two quadratics will furnish the four values of x; hence substituting for q and s their values, we get

$$x = -\frac{p}{2} + \sqrt{\left\{\frac{1}{2}\left(-\frac{p^2}{2} - a + \frac{b}{p}\right)\right\}}$$

$$x = -\frac{p}{2} - \sqrt{\left\{\frac{1}{2}\left(-\frac{p^2}{2} - a + \frac{b}{p}\right)\right\}}$$

$$x = \frac{p}{2} + \sqrt{\left\{\frac{1}{2}\left(-\frac{p^2}{2} - a - \frac{b}{p}\right)\right\}}$$

$$x = \frac{p}{2} - \sqrt{\left\{\frac{1}{2}\left(-\frac{p^2}{2} - a - \frac{b}{p}\right)\right\}};$$

where p^2 is the root of the cubic equation $z^3 + 2az^2 + (a^2 - 4c)z = b^2$.

EXAMPLES.

1. Given $x^4 + 12x = 17$ to find the four roots of the equation.

Here $a = 0$, $b = 12$, $c = -17$, and by the first method, the cubic equation in z is

$$z^3 - (\tfrac{1}{12}a^2 + c)z = \tfrac{1}{108}a^3 + \tfrac{1}{8}b^2 - \tfrac{1}{3}ac,$$

or $$z^3 + 17z = 18.$$

Hence, by inspection, $z = 1$, and therefore $r = 1, p = r + \frac{1}{3}a = 1$; $h = \sqrt{(2p - a)} = \sqrt{2}$ and $k = -\sqrt{(p^2 - c)} = -\sqrt{18} = -3\sqrt{2}$; wherefore the two quadratics are

$$x^2 - \sqrt{2}\cdot x = -1 - 3\sqrt{2}, \text{ and } x^2 + \sqrt{2}\cdot x = -1 + 3\sqrt{2},$$

which being solved give

$$x = \tfrac{1}{2}\sqrt{2} + \sqrt{(-\tfrac{1}{2} - 3\sqrt{2})}$$

$$x = \tfrac{1}{2}\sqrt{2} - \sqrt{(-\tfrac{1}{2} - 3\sqrt{2})}$$

$$x = -\tfrac{1}{2}\sqrt{2} + \sqrt{(-\tfrac{1}{2} + 3\sqrt{2})} = 1{\cdot}22748379$$

$$x = -\tfrac{1}{2}\sqrt{2} - \sqrt{(-\tfrac{1}{2} + 3\sqrt{2})} = -2{\cdot}64169735.$$

The first two of these values are imaginary.

By the second method.

Here $a = 0$, $b = 12$, $c = -17$, and the cubic equation in z is

$$z^3 + 2az^2 + (a^2 - 4c)z = b^2,$$

$$\text{or } z^3 + 68z = 144.$$

The value of z is, by inspection, easily seen to be 2, and therefore

$p = \sqrt{2}$, $s = \frac{1}{2}\left(2 + \frac{12}{\sqrt{2}}\right) = 1 + 3\sqrt{2}$, and $q = \frac{1}{2}\left(2 - \frac{12}{\sqrt{2}}\right) =$ $1 - 3\sqrt{2}$: consequently the two quadratics are

$$x^2 + \sqrt{2}\cdot x = -1 + 3\sqrt{2}, \text{ and } x^2 - \sqrt{2}\cdot x = -1 - 3\sqrt{2},$$

which are the same as in the former solution.

Find the four roots, or values of x in each of the following equations :—

	Answers.
2. $x^4 - 55x^2 - 30x + 504 = 0.$	$3, 7, -4, -6.$
3. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$	$1, 2, -2, -3.$
4. $x^4 - 8x^3 + 14x^2 + 4x = 8.$	$3+\sqrt{5}, 3-\sqrt{5}, 1+\sqrt{3}, 1-\sqrt{3}.$
5. $x^4 - 17x^2 - 20x - 6 = 0.$	$2+\sqrt{7}, 2-\sqrt{7}, -2+\sqrt{2}, -2-\sqrt{2}.$
6. $x^4 - 27x^3 + 162x^2 + 356x = 1200.$	2·05607347, − 3. 13·15306946, 14·79085207.
7. $x^4 - 12x^2 + 12x = 3.$	0·4432767, 0·6060183. 2·8580833, − 3·9073783.

RESOLUTION OF EQUATIONS

By Approximation.

Find, by trial, a number nearly equal to the root sought, which call r; and let z be made to denote the difference between this assumed root and the true root x.

Then, instead of x in the given equation, substitute its equal $r \pm z$, and there will arise a new equation involving only z and known quantities.

Reject all the terms of this equation in which z is of two or more dimensions; and the approximate value of z may then be determined by means of a simple equation.

And if the value, thus found, be added to, or subtracted from, that of r, according as r was assumed too little or too great, it will give a near value of the root required.

But as this approximation will seldom be sufficiently exact, the operation must be repeated by substituting the number thus found, for r in the abridged equation exhibiting the value of z; when a second correction of z will be obtained, which being added to, or subtracted from r, will give a nearer value of the root than the former.

And by again substituting this last number for r, in the above-mentioned equation, and repeating the same process as often as may be thought necessary, a value of x may be found to any degree of accuracy required.

Note. The decimal part of the root as found both by this and the next rule, will, in general, about double itself in extent of figures at each operation; and therefore it would be useless, as well as troublesome, to use a much greater number of figures than these in the several substitutions for the values of r.*

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* It may here be observed, that if any of the roots of an equation be whole numbers, they may be determined by substituting 1, 2, 3, 4, &c., successively,

EXAMPLES.

1. Given $x^3 + x^2 + x = 90$, to find the value of x by approximation.

Here the root, as found by a few trials, is nearly equal to 4.

Let therefore $4 = r$, and $r + z = x$, then

$$x^3 + x^2 + x = (r + z)^3 + (r + z)^2 + r + z$$

$$\left.\begin{aligned} = r^3 + 3r^2z + r^2 + 2rz + r + z \\ + 3rz^2 + z^3 + z^2 \end{aligned}\right\} = 90.$$

And by rejecting the terms involving z^2 and z^3, as small in comparison with z, we shall have

$$r^3 + r^2 + r + 3r^2z + 2rz + z = 90;$$

$$\text{whence } z = \frac{90 - r^3 - r^2 - r}{3r^2 + 2r + 1} = \frac{90 - 64 - 16 - 4}{48 + 8 + 1} = \frac{6}{57} = \cdot10,$$

and consequently $x = 4\cdot1$, *nearly*.

Again, if $4\cdot1$ be substituted in the place of r, in the last equation, we shall have

$$z = \frac{90 - r^3 - r^2 - r}{3r^2 + 2r + 1} = \frac{90 - 68\cdot921 - 16\cdot81 - 4\cdot1}{50\cdot43 + 8\cdot2 + 1} = \cdot00283.$$

And consequently $x = 4\cdot1 + \cdot00283 = 4\cdot10283$, for a *second approximate* value of x.

And if the first four figures, $4\cdot102$, of this number be again substituted

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both in *plus* and in *minus*, for the unknown quantity, till a result is obtained equal to that in the question; when those that are found to succeed will be the roots required.

Or, since the last term of any equation is always equal to the continued product of all its roots, the number of these trials may be generally diminished, by finding all the divisors of that term, and then substituting them both in *plus* and in *minus*, as before, for the unknown quantity; when those that give the proper result will be the rational roots sought; but if none of them are found to succeed, it may be concluded that the equation cannot be resolved by this method; the roots, in that case, being either irrational or imaginary.

for r, in the same equation, 4·1028325 a still nearer value of the root will be obtained; and so on, as far as may be thought necessary.

2. Given $x^2 + 20x = 100$, to find the value of x by approximation.
Ans. $x = 4{\cdot}1421356$.

3. Given $x^3 + 9x^2 + 4x = 80$, to find the value of x by approximation.
Ans. $x = 2{\cdot}4721359$.

4. Given $x^4 - 38x^3 + 210x^2 + 538x + 289 = 0$, to find the value of x by approximation. Ans. $x = 30{\cdot}53565375$.

5. Given $x^5 + 6x^4 - 10x^3 - 112x^2 - 207x - 110 = 0$, to find the value of x by approximation. Ans. 4·46410161.

APPROXIMATION BY DOUBLE POSITION.

The roots of equations, of all degrees, can also be determined, to any degree of exactness, by means of the following easy rule of Double Position; which, though it has not been generally employed for this purpose, will be found, in some respects, superior to the former, as it can be applied, at once, to any unreduced equation, consisting of surds, or compound quantities, as readily as if it had been brought to its usual form.*

Rule. Find, by trial, two numbers nearly equal to the root required, and substitute them in the given equation instead of the unknown quantity, noting the results that are obtained from each.

Then, as the difference of these results is to the difference of the two assumed numbers, so is the difference between the true result, given by the question, and either of the former, to the correction of the number belonging to the result used; which correction being added to that number when it is too little, or subtracted from it when it is too much, will give the root required, *nearly*.

And if the number thus determined, and the nearest of the two former, or any other that appears to be more accurate, be now taken as the assumed roots, and the operation be repeated as before, a new value of the unknown quantity will be obtained still more correct than the first; and so on, proceeding in this manner as far as may be judged necessary.*

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* Another method of approximating to the roots of equations will be found in the *Theory of Equations.*

EXAMPLES.

1. Given $x^3 + x^2 + x = 100$, to find an approximate value of x.

Here it is soon found, by a few trials, that the value of x lies between 4 and 5.

Hence, by taking these as the two assumed numbers, the operation will stand as follows :—

	First Sup.				*Second Sup.*		
	4	. .	x	. .	5		
	16	. .	x^2	. .	25		
	64	. .	x^3	. .	125		
	84		Results		155		
	155	. .	5	. .	100		
	84	. .	4	. .	84		
Therefore	71	:	1	: :	16	:	·225.

And consequently $x = 4 + {\cdot}225 = 4{\cdot}225$, *nearly.*

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* The above rule of Double Position, which is much more simple and commodious than the one commonly employed for this purpose, is the same as that which was first given in the octavo edition of *Bonnycastle's Arithmetic.*

To this we may further add, that when one of the roots of an equation has been found, either by this method or the former, the other roots may be determined as follows :—

Bring all the terms to the left-hand side of the equation, and divide the whole expression, so formed, by the difference between the unknown quantity (x) and the root first found; and the resulting equation will then be depressed a degree lower than the given one.

Find a root of this new equation, by approximation, as in the first instance, and the number so obtained will be a second root of the original equation.

Then, by means of this root, and the unknown quantity, depress this second equation a degree lower, and thence find a third root: and so on, till the equation is reduced to a quadratic; when the two roots of this, together with the former, will be the roots of the equation required.

Thus, in the equation $x^3 - 15x^2 + 63x = 50$, the first root is found, by approximation, to be 1·02804. Hence

$$x - 1{\cdot}02804)\ x^3 - 15x^2 + 63x - 50\ (x^2 - 13{\cdot}97196x + 48{\cdot}63627 = 0.$$

And

Again, if 4·2 and 4·3 be now taken as the two assumed numbers, the operation will stand thus :—

First Sup.		*Second Sup.*
4·2	. . x . .	4·3
17·64	. . x^2 . .	18·49
74·088	. . x^3 . .	79·507
95·928	Results	102·297
102·297	. . 4·3 . .	102·297
95·928	. . 4·2 . .	100

Therefore 6·369 : ·1 :: 2·297 : ·036.

And consequently $x = 4·3 - ·036 = 4·264$, *nearly.*

Again, let 4·264 and 4·265 be the two assumed numbers ; then

First Sup.		*Second Sup.*
4·264	. . x . .	4·265
18·181696	. . x^2 . .	18·190225
77·526752	. . x^3 . .	77·581310
99·972448	Results	100·036535
	Therefore	
100·036535	4·265	100
99·972448	4·264	99·972448

·064087 : ·001 :: ·027552 : ·0004299.

And consequently

$x = 4·264 + ·0004299 = 4·2644299$, *very nearly.*

2. Given $(\frac{1}{5}x^2 - 15)^2 + x\sqrt{x} = 90$, to find an approximate value of x.

Here, by a few trials, it will be soon found that the value of x lies between 10 and 11 ; which let, therefore, be the two assumed numbers, agreeably to the directions given in the rule.

And the two roots of the quadratic equation, $x^2 - 13·97196x = -48·63627$, found in the usual way, are 6·57653 and 7·39543.

So that the three roots of the given cubic equation, $x^3 - 15x^2 + 63x = 50$, are 1·02804, 6·57653, and 7·39543; their sum being = 15, the coefficient of the second term of the equation, as it ought to be when they are right.

Then

	First Sup.				Second Sup.	
	25	. .	$(\frac{1}{5}x^2 - 15)^2$	. .	84·64	
	31·623	. .	$x\sqrt{x}$	. .	36·483	
	56·623		Results		121·123	
	121·123	. .	11	. .	121·123	
	56·623	. .	10	. .	90	
Hence	64·5	:	1	: :	31·123	: ·482.

And consequently $x = 11 - {\cdot}482 = 10{\cdot}518$.

Again, let 10·5 and 10·6 be the two assumed numbers, then

First Sup.				Second Sup.	
49·7025	. .	$(\frac{1}{5}x^2 - 15)^2$	. .	55·830784	
34·0239	. .	$x\sqrt{x}$	. .	34·511099	
83·7264	. .	Results	. .	90·341883	
		Hence			
90.341883	. .	10·6	. .	90·341883	
83·7264	. .	10·5	. .	90·	
6.615483	:	·1	: :	·341883	: ·0051679.

And consequently

$x = 10{\cdot}6 - {\cdot}0051679 = 10{\cdot}5948321$, *very nearly*.

EXAMPLES FOR PRACTICE.

1. Given $x^3 + 10x^2 + 5x = 2600$, to find a near approximate value of x.
Ans. $x = 11\ 00679$.

2. Given $2x^4 - 16x^3 + 40x^2 - 30x + 1 = 0$, to find a near value of x.
Ans. $x = 1{\cdot}284724$.

3. Given $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$, to find the value of x.
Ans. 8·414455.

4. Given $\sqrt[3]{(7x^3 + 4x^2)} + \sqrt{(20x^2 - 10x)} = 28$, to find the value of x.
Ans. 4·510661.

5. Given $\sqrt{\{144x^2 - (x^2 + 20)^2\}} + \sqrt{\{196x^2 - (x^2 + 24)^2\}} = 114$, to find the value of x.
Ans. 7·123883.

RATIOS, PROPORTION, AND VARIATION.

I. Ratios.

Definition 1. *Ratio* is the relation which one quantity has to another of the same kind. The comparison is made by considering what multiple, part or parts, one of the quantities is of the other.

Thus, if a and b be two quantities of the same kind, the fraction $\frac{a}{b}$ will express the ratio of a to b. The two quantities which a ratio compares are called the *terms* of the ratio, and the former term a is called the *antecedent*, and the latter term b the *consequent* of the ratio.

2. A ratio of *equality* is one in which the antecedent is *equal* to the consequent; a ratio of *greater inequality* or *majority* is one in which the antecedent is *greater* than the consequent; and a ratio of *less inequality* or *minority* is one in which the antecedent is *less* than the consequent.

By means of the preceding mode of representation of a ratio, we are enabled to make comparisons between two or more ratios. Thus

$$\text{the ratio of 3 to 4} = \tfrac{3}{4}, \text{ and the ratio of 5 to 6} = \tfrac{5}{6};$$

but $\frac{5}{6}$ is greater than $\frac{3}{4}$; and therefore the ratio of 5 to 6 is *greater* than the ratio of 3 to 4, or the ratio of 3 to 4 is *less* than the ratio of 5 to 6.

In a similar manner the ratio of a to b is greater or less than the ratio of c to d according as $\frac{a}{b}$ is greater or less than $\frac{c}{d}$, or as $\frac{ad}{bd} >$ or $< \frac{bc}{bd}$, or as ad is greater or less than bc.

Hence the relative values of numerical ratios, when expressed fractionally, may be readily known by reducing them to a common denominator; the numerators will then express their relative values.

A comparison of these will indicate whether any two are in a ratio of majority, equality, or minority.

Prop. I. *A ratio of greater inequality is diminished, and one of less inequality is increased by adding the same or equal quantities to each of its terms.*

Let $\frac{a}{b}$ be the given ratio, and $\frac{a+x}{b+x}$ the new ratio, in which x is added to each of its terms; then, reducing these fractions to a common denominator, we have

$$\frac{a+x}{b+x} \text{ and } \frac{a}{b} = \frac{ab+bx}{b(b+x)} \text{ and } \frac{ab+ax}{b(b+x)}.$$

But $ab + ax > ab + bx$ if $a > b$, and $ab + ax < ab + bx$, if $a < b$; hence

$$\text{if } a > b,\ \frac{a+x}{b+x} < \frac{a}{b}, \text{ or the ratio is } \textit{diminished};$$

$$\text{but if } a < b,\ \frac{a+x}{b+x} > \frac{a}{b}, \text{ or the ratio is } \textit{increased.}$$

If the ratio be one of equality, and if the same or equal quantities be added to each of its terms, the new ratio will still be one of equality.

For if $a = b$, then $a+x = b+x$, and the ratio $\frac{a+x}{b+x}$ is still one of equality, and equal therefore to the original ratio $\frac{a}{b}$.

Prop. II. *A ratio of greater inequality is increased, and one of less inequality is diminished by subtracting the same or equal quantities from each of its terms.*

Let $\frac{a}{b}$ be the given ratio, and $\frac{a-x}{b-x}$ the new ratio, where x is less than either a or b; then as before

$$\frac{a-x}{b-x} \text{ and } \frac{a}{b} = \frac{ab-bx}{b(b-x)} \text{ and } \frac{ab-ax}{b(b-x)}.$$

Now, if a is greater than b, $ab - bx$ is *greater* than $ab - ax$, for more is subtracted from ab in the latter case than in the former; but if a is less than b, then $ab-bx$ is *less* than $ab-ax$, since more is subtracted from ab in the former case than in the latter; hence the truth of the proposition is obvious.

Def.—If the antecedents of two or more ratios be multiplied together, and their consequents be multiplied together, the new ratio is said to be *compounded* of the other ratios, and it is called the ***compound ratio*** of these several ratios.

Thus if the ratios be

$$a : b; \quad c : d; \quad e : f; \quad g : h;$$

then their compound ratio is $aceg : bdfh$, or $\frac{aceg}{bdfh}$.

When *equal* ratios $a : b$, $a : b$, $a : b$, *etc.* are compounded, the resulting ratios are called the double or *duplicate ratio*, the triple or *triplicate* ratio, *etc.* of the given ratio, according as the number of equal ratios which are compounded is *two*, *three*, etc.

Thus, if the ratio $a : b$ be compounded with the ratio $a : b$; then $a^2 : b^2$ is the duplicate ratio of $a : b$, and $a^3 : b^3$ is the triplicate ratio of $a : b$, because it is compounded of *three* ratios, each equal to the ratio $a : b$.

In like manner $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called the half or *subduplicate* ratio of $a : b$; $a^{\frac{1}{3}} : b^{\frac{1}{3}}$, the *subtriplicate* ratio, and $a^{\frac{3}{2}} : b^{\frac{3}{2}}$ the *sesquiplicate* ratio of $a : b$; the latter ratio being compounded of $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ and $a : b$, the simple and subduplicate ratios of $a : b$.

PROP. III. *If there be any number of ratios such that the consequent of one ratio is always the antecedent of the succeeding ratio, the compound ratio is equal to the ratio of the first antecedent to the last consequent.*

Let the proposed ratios be

$$\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{e}, \frac{e}{f}, \frac{f}{g}, \frac{g}{h}, \text{etc.}$$

then their compound ratio is

$$\frac{abcdefg \ldots\ldots}{bcdefgh \ldots\ldots} = \frac{a}{h}.$$

II. PROPORTION.

Definition 1. Proportion is the equality of ratios.

If $a:b$ and $c:d$ be two ratios, and $\frac{a}{b}=\frac{c}{d}$; then the four quantities or magnitudes, a, b, c, d, are said to be proportional, and usually written

$$a:b::c:d.$$

2. The *extreme* terms are a and d, and the *mean* terms b and c.

3. When the second and third terms of a proportion are identical, as

$$a:b::b:c,$$

then b is said to be a *mean proportional* between the extremes a and c, and c is a *third proportional* to a and b.

4. Also, if in a series of proportional quantities, the consequent of one ratio be always the antecedent of the succeeding ratio, these quantities are said to be in *continued proportion.*

Thus, if $a:b::b:c$, $b:c::c:d$, $c:d::d:e$;

then a, b, c, d, e are in continued proportion.

PROP. I. *If four quantities be proportionals, the product of the extremes will be equal to the product of the means.*

Let $a:b::c:d$; then shall $ad=bc$.

For since $\frac{a}{b}=\frac{c}{d}$, if these equals be multiplied by bd, we shall have

$$\frac{a}{b}\times bd=\frac{c}{d}\times bd, \text{ or } ad=bc.$$

PROP. II. *If the product of two quantities be equal to the product of two others, then will these four quantities be proportionals.*

Let $ad=bc$, then dividing these equals by bd, we get

$$\frac{ad}{bd}=\frac{bc}{bd} \text{ or } \frac{a}{b}=\frac{c}{d}, \text{ or } a:b::c:d.$$

PROP. VIII. *If four quantities be proportionals, then the same powers or roots of these quantities will also be proportionals.*

Let $a : b :: c : d$; then will $a^m : b^m :: c^m : d^m$.

For, since $\frac{a}{b} = \frac{c}{d}$; therefore $\left(\frac{a}{b}\right)^m = \left(\frac{c}{d}\right)^m$, or $\frac{a^m}{b^m} = \frac{c^m}{d^m}$,

where m may be either integral or fractional.

PROP. IX. *If there be any number of quantities more than two, and as many others, which taken two and two, either in order, or in a cross order, are proportionals, the first shall have to the last of the first quantities the same ratio that the first has to the last of the others.*

Let a, b, c, d, ... be any number of quantities, and p, q, r, s, . . . as many others; then if these quantities are proportionals when taken two and two in order,

$$\frac{a}{b} = \frac{p}{q},\ \frac{b}{c} = \frac{q}{r},\ \frac{c}{d} = \frac{r}{s};\ etc.;$$

$$\text{therefore } \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{p}{q} \times \frac{q}{r} \times \frac{r}{s}, \text{ or } \frac{a}{d} = \frac{p}{s}.$$

But if these quantities are proportionals when taken two and two in a cross order, then

$$\frac{a}{b} = \frac{r}{s},\ \frac{b}{c} = \frac{q}{r},\ \frac{c}{d} = \frac{p}{q},\ etc.:$$

$$\text{therefore } \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{r}{s} \times \frac{q}{r} \times \frac{p}{q}, \text{ or } \frac{a}{d} = \frac{p}{s}.$$

PROP. X. *If any number of quantities are proportionals, as one of the antecedents is to its consequent, so are all the antecedents taken together, to all the consequents.*

Let $a : b :: c : d :: e : f :: g : h$; then $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$;

therefore $ad = bc$, $af = be$, $ah = bg$, and also $ab = ba$; hence

$$ab + ad + af + ah = ba + bc + be + bg,$$

or, $a(b + d + f + h) = b(a + c + e + g)$; therefore (Prop. II.),

$$a : b :: a + c + e + g : b + d + f + h.$$

Prop. XI. *To find a mean proportional between two given quantities.*

Let a and b be the given quantities, and x the mean proportional between them; then $a : x :: x : b$; hence (Prop. I.) $x^2 = ab$,

$$\therefore\ x = \sqrt{(ab)} = \text{mean proportional sought.}$$

Prop. XII. *If four quantities be proportionals, the greatest and least together are greater than the other two together.*

If $a : b :: c : d$, and if a be the greatest, then d will be the least, and $a + d$ will be greater than $b + c$.

For since $a : b :: c : d$; therefore alternately $a : c :: b : d$, or

$$\frac{a}{c} = \frac{b}{d}; \text{ but } a > c; \text{ hence } b > d.$$

Again, $a : a - b :: c : c - d$, or $a : c :: a - b : c - d$; but

$$a > c,\ \therefore\ a - b > c - d, \text{ or } a + d > b + c.$$

III. Variation.

Variation, in this branch of Algebra, implies change of quantity or magnitude, and a quantity is said to vary as one or more others when they are so related, that every change in the latter produces a corresponding change in the former.

Def. 1. One quantity is said to vary *directly* as another, when the changed value of the former has to the changed value of the latter, the same ratio as the unchanged quantities.

Def. 2. One quantity is said to vary *inversely* as another, when the changed value of the former, multiplied by the changed value of the latter, is equal to the product of the unchanged quantities.

Def. 3. One quantity is said to vary as two others *jointly*, when it varies directly as their product; and one quantity is said to vary as any number of others, when it varies directly as their continued product.

Def. 4. One quantity is said to vary *directly* as a second quantity, and *inversely* as a third, when it varies directly as the quotient of the second quantity divided by the third.

The symbol made use of to signify variation is $\propto$ (*varies as*); and the different kinds of variation are thus distinguished :—

	written thus :
(1) A varies *directly* as B, ...	$A \propto B$
(2) A varies *inversely* as B, ...	$A \propto \frac{1}{B}$
(3) A varies as B and C *jointly*, . . .	$A \propto B \times C$
(4) A varies as B *directly* and as C *inversely*, .	$A \propto \frac{B}{C}$.

The expressions $A \propto B$, $A \propto \frac{1}{B}$, *etc.* always imply the proportionality of four quantities ; thus if A_1 be the changed value of A corresponding to B_1 the changed value of B ; then $A : B :: A_1 : B_1$, and so of the others.

The following familiar examples will convey to the student a clear idea of the meaning of the term variation :—

(1) A schoolmaster proposes to distribute a number of oranges equally amongst a certain number of his pupils ; now if

n = the number of oranges,
p = the number of pupils,
s = the share of each pupil ;

then $n = sp$, since the whole number of oranges will be equal to the share of each multiplied by the number of pupils to whom they are to be distributed. Let n_1 denote any other number of oranges, and s_1 the share of each pupil ; then as before $n_1 = s_1 p$, the number of pupils remaining the same ; hence, from these two equations,

$$\frac{s}{n} = \frac{s_1}{n_1} = \frac{1}{p} = \text{a constant ratio ;}$$

$$\therefore\ s \propto n, \text{ by } \textit{Def.}\ 1\ ;$$

that is, the share of each pupil varies *directly* as the number of oranges to be distributed.

Again, let the number of oranges remain constant, but let the number of pupils amongst whom they are to be distributed vary ; then if p_1 = the number of pupils, and s_1 = the share of each ; then, as before,

$$n = sp,\ n = s_1 p_1,\ \therefore\ sp = s_1 p_1 = \text{constant product ;}$$

hence, (*Def.* 2), $s \propto \frac{1}{p}$, that is, the share of each pupil varies *inversely* as the number of pupils.

Since $n = sp = s \div \frac{1}{p}$, and $n = s_1 p_1 = s_1 \div \frac{1}{p_1}$, $\therefore s \div \frac{1}{p} = s_1 \div \frac{1}{p_1}$;

and therefore (*Def.* 1.) $s \propto \frac{1}{p}$, as before.

(2) Let A, B, P denote the area, base, and perpendicular of a triangle; and A_1, B_1, P_1 the area, base, and perpendicular of another triangle; then by the principles of mensuration of surfaces, we have

$$A = \tfrac{1}{2} B P \text{ and } A_1 = \tfrac{1}{2} B_1 P_1 ;$$

$$\therefore A : A_1 :: \tfrac{1}{2} B P : \tfrac{1}{2} B_1 P_1 :: B P : B_1 P_1 ;$$

hence $A_1 : B_1 P_1 :: A : B P$, by Prop. VI., (1) p. 144;

$$\therefore A \propto B P ;$$

that is (*Def.* 3), the area of a triangle varies as the base and perpendicular *jointly*.

Again, since $B = \frac{2A}{P}$ and $B_1 = \frac{2A_1}{P_1}$, we have

$$B_1 : B :: \frac{2A_1}{P_1} : \frac{2A}{P} :: \frac{A_1}{P_1} : \frac{A}{P} ;$$

$\therefore B_1 : \frac{A_1}{P_1} :: B : \frac{A}{P}$, Prop. VI., (1) p. 144;

$$\therefore B \propto \frac{A}{P} ;$$

that is (*Def.* 4), the base of a triangle varies as the area *directly*, and the perpendicular *inversely*.

The *Theory of Variation* is contained in the following propositions:—

PROP. I. *If* $A \propto B$, *then will* $A = mB$, *where* m *is a constant quantity.*

Let A_1 and B_1 be corresponding values of A and B; then since $A \propto B$,

$$A : B :: A_1 : B_1 \text{ or } \frac{A}{B} = \frac{A_1}{B_1} = \text{constant quantity, } (\textit{Def. } 1).$$

If $\frac{A_1}{B_1} = m$; then $\frac{A}{B} = m$, and $A = mB$.

Conversely, if $A = mB$, then $A \propto B$.

Prop. II. *If* $A \propto B$ *and* $B \propto C$; *then* $A \propto C$.

For (Prop. I.) $A = mB$, $B = nC$; therefore $A = mnC$; hence

$$A \propto C.$$

Also, if $A \propto B$ and $B \propto \frac{1}{C}$; then $A \propto \frac{1}{C}$. This is proved in a similar manner.

Prop. III. *If* $A \propto \frac{1}{B}$ *and* $B \propto \frac{1}{C}$; *then* $A \propto C$.

For since $A = \frac{m}{B}$ and $B = \frac{n}{C}$; therefore $A = m \div \frac{n}{C} = \frac{m}{n} C$;

hence $A \propto C$.

Prop. IV. *If* $A \propto C$ *and* $B \propto C$; *then will* $A \pm B \propto C$.

For $A = mC$ and $B = nC$; therefore $A \pm B = (m \pm n) C$; that is

$$A \pm B \propto C.$$

Prop. V. *If* $A \propto C$ *and* $B \propto C$; *then will* $AB \propto C^2$ *and* $\sqrt{AB} \propto C$.

For $A = mC$ and $B = nC$; then $AB = mnC^2$ and $\sqrt{AB} = \sqrt{mn} \,.\, C$;

$$\therefore \sqrt{}AB \propto C^2, \text{ and } \sqrt{AB} \propto C.$$

Prop. VI. *If* $A \propto BC$ *then will* $B \propto \frac{A}{C}$ *and* $C \propto \frac{A}{B}$.

For $A = mBC$; hence $B = \frac{1}{m} \,.\, \frac{A}{C}$ and $C = \frac{1}{m} \,.\, \frac{A}{B}$; therefore

$$B \propto \frac{A}{C} \text{ and } C \propto \frac{A}{B}.$$

Prop. VII. *If* $A \propto B$ *and* $C \propto D$; *then will* $AC \propto BD$ *and* $\frac{A}{C} \propto \frac{B}{D}$.

For $A = mB$ and $C = nD$; therefore $AC = mnBD$ and $\frac{A}{C} = \frac{m}{n} \,.\, \frac{B}{D}$;

$$\therefore AC \propto BD \text{ and } \frac{A}{C} \propto \frac{B}{D}.$$

PROP. VIII. *If three quantities be so related that when either the second or third is constant, the first varies as the other; then when neither are constant, the first will vary as their product.*

If $A \propto B$ while C is constant, and $A \propto C$ while B is constant; then when neither B nor C is constant, $A \propto BC$.

For if $A \propto B$, then $A = mB$; wherefore the expression to which A is equal must involve B as a simple factor. Again, if $A \propto C$, then $A = nC$, and the expression which is equal to A must also involve C as a simple factor; hence A involves BC, and factors independent of these; therefore

$$A = pBC, \text{ or } A \propto BC.$$

Cor. If $A \propto B$ when C and D are invariable, and $A \propto C$, when B and D are invariable, and $A \propto D$ when B and C are invariable; then

$$A \propto BCD.$$

This proposition contains the whole doctrine of *Proportion*, and affords simple solutions to all questions in that important branch of arithmetic.

For suppose a certain effect is to be produced, or work (W) to be done by a number of persons (N) in the time (T); then it is evident that the work done will vary as the number of persons (N) when the time (T) is the same, and it will likewise vary as the time (T) when the number (N) of workmen remains the same; hence by the proposition

$$W \propto NT$$

and if w, n, t be corresponding values of W, N, T, then

$$W : w :: NT : nt.$$

Ex. If 120 men in 3 days of 12 hours each, can dig a trench 30 yards long, 2 broad and 4 deep; how many men must be employed to dig a trench 50 yards long, $1\frac{1}{2}$ broad and 2 deep, in 9 days of 15 hours each?

Here $W = 30 \times 2 \times 4$, $w = 50 \times 1\frac{1}{2} \times 2$, $T = 3 \times 12$, $t = 9 \times 15$; and $N = 120$; but

$$W : w :: NT : nt;$$

$$\therefore 30 \times 2 \times 4 : 50 \times 1\tfrac{1}{2} \times 2 :: 120 \times 3 \times 12 : n \times 9 \times 15$$

$$\text{hence } n = \frac{50 \times 1\frac{1}{2} \times 2 \times 120 \times 3 \times 12}{30 \times 2 \times 4 \times 9 \times 15} = 20 \text{ men.}$$

EXAMPLES IN RATIOS, PROPORTION, AND VARIATION.

1. Determine which is the greater ratio, $a+2b : a+b$ or $a+3b : a+2b$.
The fractions expressing these ratios, reduced to a common denominator, are

$$\frac{a^2+4ab+4b^2}{(a+b)(a+2b)} \text{ and } \frac{a^2+4ab+3b^2}{(a+b)(a+2b)};$$

the former of which is greater than the latter; hence

the ratio $a+2b : a+b$ is greater than the ratio $a+3b : a+2b$.

2. If $a : b :: b : c$, then $a^2 - b^2 : a :: b^2 - c^2 : c$.

$$\text{For } \frac{a}{b} = \frac{b}{c}, \therefore \frac{a^2}{b^2} = \frac{b^2}{c^2} \text{ and } \frac{a^2}{b^2} - 1 = \frac{b^2}{c^2} - 1, \text{ or } \frac{a^2-b^2}{b^2} = \frac{b^2-c^2}{c^2};$$

but $b^2 = ac$ (Prop. III. p. 144); therefore $\frac{b^2}{a} = c$, and

$$\frac{a^2-b^2}{b^2} \times \frac{b^2}{a} = \frac{b^2-c^2}{c^2} \times c \text{ or } \frac{a^2-b^2}{a} = \frac{b^2-c^2}{c}.$$

3. Given that $y^2 \propto x^2 - a^2$, and when $x^2 = a^2 + b^2$, then $y = \frac{b^2}{a}$; find the equation between x and y.

Here $y^2 = m(x^2 - a^2)$, and writing in this $\frac{b^2}{a}$ for y and $a^2 + b^2$ for x^2 we have

$$\frac{b^4}{a^2} = m(a^2 + b^2 - a^2) = mb^2, \therefore m = \frac{b^2}{a^2},$$

$$\text{and } y^2 = \frac{b^2}{a^2}(x^2 - a^2).$$

4. Of the ratios $a + b : a - b$, and $a^2 + b^2 : a^2 - b^2$, which is the greater, supposing a greater than b. Ans. $\frac{a+b}{a-b}$.

5. Find the ratio compounded of the ratios $3 : 4$, $8 : 15$ and $5 : 4$.
Ans. $1 : 2$.

6. If $x : y$ in the duplicate ratio of $a : b$, and $a : b$ in the subduplicate ratio of $a + x : a - y$; shew that

$$2x : a :: x - y : y.$$

7. If $a + b : c + d :: c - d : a - b$; then prove that $a + c : b + d :: b - d : a - c$.

8. If $a : b :: c : d$, shew that $(a + d) - (b + c) = \frac{(a - b)(a - c)}{a}$.

9. If $y =$ the sum of two quantities, one of which varies as x, and the other varies as x^2, and when $x = 1$, $y = 14$, when $x = 2$, $y = 46$, find the equation between y and x. Ans. $y = 5x + 9x^2$.

10. Given that $z \propto x + y$, that $y \propto x^2$, and that when $x = 1$, $y = 2$ and $z = 3$; find the equation between z and x.
Ans. $z = x + 2x^2$.

11. If $y = p + q + r$, where p is invariable, $q \propto x$ and $r \propto x^2$, and when $x = 1, 2, 3$, the simultaneous values of y are 2, 3, 6 respectively; find the value of y when $x = 5$. Ans. $y = 18$.

12. A and B are two vessels, each containing a mixture of water and wine, A in the ratio of $2 : 3$, B in the ratio $3 : 7$; what quantity must be taken from each to form a third mixture which shall contain 5 gallons of water and 11 of wine. Ans. 2 gallons from A, and 14 from B.

13. The hour and minute hands of a watch are together at 12 o'clock, what is the exact time when they are in conjunction between 3 and 4 o'clock. Ans. $3\frac{3}{11}$ hours.

14. The weight of a spherical shell is $\frac{7}{8}$ of the weight of a solid sphere of the same substance and radius; find the ratio of the radii of the interior and exterior surfaces of the shell, having given that the weight of a sphere varies as the cube of its diameter. Ans. $1 : 2$.

15. A diamond of a carats is worth m times a ruby of b^2 carats, and both together are worth c pounds; find the value of a diamond and a ruby, each weighing w^2 carats, having given that the value of diamonds varies as the square of their weight, and the square of the value of rubies varies as the cube of their weight.

Ans. Value of the diamond is $\frac{mcw^4}{(m + 1)a^2}$ pounds,

and value of the ruby is $\frac{cw^3}{(m + 1)b^3}$ pounds.

ARITHMETICAL PROGRESSION.

An *arithmetical series*, or an *arithmetical progression*, is a series of quantities of which the difference between every two consecutive terms is the *same*. Thus

$$2,\ 5,\ 8,\ 11,\ etc.;\ \text{and}\ 25h,\ 21h,\ 17h,\ 13h,\ etc.,$$

are in arithmetical progression, the former being an increasing series whose common difference is 3, and the latter a decreasing series whose common difference is $-4h$.

If $a =$ the first term of the series, $d =$ the common difference between the terms, $n =$ the number of terms, $l =$ the last term, and $s =$ the sum of the series; then the *second* term of the series $= a+d$, the *third* $= a + 2d$, the *fourth* $= a + 3d$, and the nth or *last* term $= a + (n - 1)d$; hence

$$l = a + (n - 1)d \quad . \ . \ . \ . \ . \quad (1)$$

Again, if

$$s = a + (a+d) + (a+2d) + \ldots\ldots + (l-2d) + (l-d) + l;$$

$$\text{then } s = l + (l-d) + (l-2d) + \ldots\ldots + (a+2d) + (a+d) + a;$$

$$\therefore 2s = (a+l)+(a+l)+(a+l) + \ldots\ldots + (a+l)+(a+l)+(a+l)$$

$$= (a+l).n, \text{ since there are } n \text{ equal terms;}$$

$$\therefore s = \left(a + l\right)\frac{n}{2}. \quad . \ . \ . \ . \ . \ . \quad (2)$$

Writing the value of l from equation (1) in (2) gives

$$s = \left\{2a + (n - 1)d\right\}\frac{n}{2} \quad . \ . \ . \ . \quad (3).$$

The formulæ (1) and (3) are sufficient for the solution of all questions which can be proposed in this branch of Algebra, and comprise the entire *Theory* of Arithmetical Progression. From these, however, more simple expressions for particular cases may be derived.

Cor. 1. The sum of n terms of the natural numbers 1, 2, 3, 4, 5, &c.

$$s = \tfrac{1}{2}n(n + 1).$$

Cor. 2. The sum of n terms of the even numbers 2, 4, 6, 8, 10, &c.

$$s = n(n + 1).$$

Cor. 3. The sum of n terms of the odd numbers, 1, 3, 5, 7, 9, &c.

$$s = n^2.$$

Hence also, if, of the four quantities a, d, n, and s, any three be known; by a transformation of the preceding equations, the fourth may be found from a solution of the new equation so obtained.

An *arithmetical mean* is the term, or *arithmetical means* are the terms which intervene between any two terms of an arithmetical series. Thus in the series 3, 4, 5, the term 4 is an arithmetical mean between 3 and 5, and in the series 3, 4, 5, 6, the terms 4 and 5 are arithmetical means between the extreme terms 3 and 6.

To insert any number of arithmetical means between two given numbers a and l.

Let n denote the number of means; then $n + 2$ will be the number of terms inclusive of a and l; hence writing $n + 2$ for n in formula (1) we get

$$l = a + (n + 1)d, \text{ and } \therefore\ d = \frac{l - a}{n + 1} \quad . \ . \ . \ (4)$$

This common difference, being added to a will give the first mean, and being added to the first mean will give the second mean, and so on.

EXAMPLES FOR EXERCISE.

1. Find the sum of 20 terms of the series 3, 5, 7, 9, *etc.*

Here $a = 3$, $d = 2$, and $n = 20$; therefore

$$s = \left\{2a + (n - 1)d\right\}\frac{n}{2} = \left(6 + 19 \times 2\right)\frac{20}{2}$$

$$= (6 + 38) \times 10 \qquad = 44 \times 10 = 440 = \text{sum required.}$$

2. The *fifth* term of an arithmetic series is 84, and the *tenth* term is 44; find the 20th term, and the sum of 25 terms of the series.

The *fifth* term $= a + (5 - 1)d = a + 4d = 84,$

the *tenth* term $= a + (10 - 1)d = a + 9d = 44.$

Subtracting the former from the latter of these equations, gives

$$5d = -40, \text{ or } d = -8 = \text{common difference.}$$

Hence $a = 84 - 4d = 84 + 32 = 116 =$ the first term.

Again, the 20th term $= a + (20 - 1)d = 116 - 19 \times 8.$

$$= 116 - 152 \qquad = -36.$$

And for the sum of 25 terms, we get

$$s = \left\{2a + (n - 1)d\right\}\frac{n}{2} = \left(2 \times 116 - 24 \times 8\right)\frac{25}{2}$$

$$= \left(232 - 192\right)\frac{25}{2} \quad = 20 \times 25 = 500.$$

3. Insert 5 arithmetical means between 3 and 27.

Here $a = 3$, $l = 27$, and $n = 7$ · therefore by formula (1)

$$27 = 3 + (7 - 1)d, \text{ or } d = \frac{27 - 3}{6} = 4;$$

hence the 5 means are 7, 11, 15, 19, 23.

4. The first term of an arithmetical series is 100, the common difference -3, and the number of terms 34; find the sum of the series.
Ans. 1717.

5. Find the sum of n terms of the natural numbers 1, 2, 3, *etc.*
Ans. $\frac{n(n + 1)}{2}$.

6. Find the sum of n terms of the odd numbers 1, 3, 5, *etc.*
Ans. n^2.

7. Find the arithmetical mean between the two quantities a and b.
Ans. $\frac{a + b}{2}$.

8. Shew that in any arithmetical series, the sum of the two extremes is equal to the sum of any two terms equally distant from them, or to twice the middle term, when the number of terms is odd.

9. How many strokes does the clock at Venice, which goes on to 24 o'clock, strike in a day? Ans. 300.

10. Find the 365th term of the series 2, 4, 6, 8, *etc.*
Ans. 730.

11. The first term of a series is 10, and the common difference $-\frac{1}{3}$; find the sum of 21 terms. Ans. 140.

12. Insert 5 arithmetical means between 2 and 20.
Ans. 5, 8, 11, 14, 17.

13. There are n arithmetical means inserted between 1 and 31, and the seventh is to the $(n-1)$th as 5 to 9; find the number of means.
Ans. 14.

14. The nth term of an arithmetical series is $\frac{3n-1}{6}$; find the sum of n terms. Ans. $\frac{n^2}{4}+\frac{n}{12}$.

15. The first term of an arithmetical series is n^2-n+1, and the common difference 2; shew that the sum of n terms $=n^3$, and also that

$$1^3=1,\ 2^3=3+5,\ 3^3=7+9+11;\ \textit{etc.}$$

16. The *fifth* and *ninth* terms of an arithmetical progression are 19 and 35; what is the seventh term, and the sum of 12 terms?
Ans. 27, and 300.

17. One hundred stones being placed on the ground, in a straight line, at the distance of a yard from each other; how far will a person travel, who shall bring them one by one to a basket, placed at a distance of a yard from the first stone? Ans. 5 miles and 1300 yards.

18. A debt can be discharged in a year by paying one pound the first week, three the second, five the third, &c.; required the amount of the debt. Ans. L.2704.

19. A general ranging a division of his army in the form of a solid equilateral triangle, finds he has 50 men in excess, but increasing the side by one man, has a deficiency of 51 men to fill up the triangle; how many men had he? Ans. 5100 men.

20. From two towns, 168 miles distant from each other, two persons, A and B, set out to meet each other: A goes 3 miles the first day, 5 the second, 7 the third, and so on; B goes 4 miles the first day, 6 the second, 8 the third, and so on; in how many days after starting will they meet?
Ans. In 8 days.

21. Find four numbers in arithmetical progression such, that the product of the extremes shall be 45, and the product of the means 77.
Ans. 3, 7, 11, 15.

22. Find four numbers in arithmetical progression such, that their common difference shall be 4, and their continued product 176985.

Ans. 15, 19, 23, 27.

23. If $s_1, s_2, s_3, \ldots . s_p$ denote the respective sums of p arithmetical progressions, each continued to n terms, of which $1, 2, 3, \ldots . p$ are the respective first terms, and $1, 3, 5, \ldots . 2p-1$ are the common differences; prove that

$$s_1 + s_2 + s_3 + \ldots . s_p = (np+1)\frac{np}{2}.$$

GENERAL FORMULA

For determining the sum of n terms of a series of which the terms are the products of the corresponding terms of two arithmetical series, having the same common difference.

Let $t_1, t_2, t_3, \ldots . t_n$ be the several terms of an arithmetical series whose first term is a, and common difference d, and

$t'_1, t'_2, t'_3, \ldots . t'_n$ those of another arithmetical series whose first term is b, and common difference d; and let it be required to find the sum of n terms of the following series, or the value of s, when

$$s = t_1t'_1 + t_2t'_2 + t_3t'_3 + \ldots \ldots + t_nt'_n.$$

$$\text{Let } s_1 = t_1 + t_2 + t_3 + \ldots . t_n = \{2a + (n-1)d\}\tfrac{n}{2} \ldots . (1)$$

$$s_2 = t'_1 + t'_2 + t'_3 + \ldots . t'_n = \{2b + (n-1)d\}\tfrac{n}{2} \ldots . (2),$$

$$\text{where } t_1 = a,\ t_2 = a + d,\ t_3 = a + 2d, \ldots . t_n = a + (n-1)d,$$

$$t'_1 = b,\ t'_2 = b + d,\ t'_3 = b + 2d, \ldots . t'_n = b + (n-1)d;$$

$$\therefore t'_n - t_n = b - a \ldots\ldots\ldots\ldots\ldots . (3).$$

Now the $(n+1)$th terms of the series (1) and (2) are

$$t_{n+1} = t_n + d = a + nd, \text{ and } t'_{n+1} = t'_n + d = b + nd;$$

$$\therefore t^2_{n+1} \times t'_{n+1} = (t_n + d)^2(t'_n + d)$$

$$= t^2_nt'_n + 2dt_nt'_n + dt^2_n + d^2(t'_n + 2t_n) + d^3.$$

But from (3) $t_n = t'_n - (b-a)$; $\therefore dt^2_n = dt_n\{t'_n - (b-a)\} = dt_nt'_n - d(b-a)t_n$;

hence the last equation is transformed to

$$t^2_{n+1} \times t'_{n+1} = t^2_n t'_n + 3dt_n t'_n + d\{2d-(b-a)\}t_n + d^2t'_n + d^3;$$

or if we put $p = 2d-(b-a)$, we shall have

$$t^2_{n+1} \times t'_{n+1} = t^2_n t'_n + 3dt_n t'_n + pdt_n + d^2t'_n + d^3 \quad \ldots\ldots (4).$$

Now, if in the equation (4) n be expounded successively by the series of natural numbers 1, 2, 3, ... n, we shall have, when

$$\begin{aligned}
n = 1,\ t^2_2 t'_2 \quad &= t^2_1 t'_1 + 3dt_1 t'_1 + pdt_1 + d^2t'_1 + d^3 \\
n = 2,\ t^2_3 t'_3 \quad &= t^2_2 t'_2 + 3dt_2 t'_2 + pdt_2 + d^2t'_2 + d^3 \\
n = 3,\ t^2_4 t'_4 \quad &= t^2_3 t'_3 + 3dt_3 t'_3 + pdt_3 + d^2t'_3 + d^3 \\
\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n = n,\ t^2_{n+1} \times t'_{n+1} &= t^2_n t'_n + 3dt_n t'_n + pdt_n + d^2t'_n + d^3.
\end{aligned}$$

Taking the sum of these vertical columns, and cancelling the like terms on each side, we have at once

$$t^2_{n+1} \times t'_{n+1} = t^2_1 t'_1 + 3ds + pds_1 + d^2s_2 + nd^3;$$

$$\therefore\ s = \frac{t^2_{n+1} \times t'_{n+1} - t^2_1 t'_1 - pds_1 - d^2s_2 - nd^3}{3d} \quad \ldots (5)$$

Substituting in (5) the values of t_{n+1}, t'_{n+1}, t_1, t'_1, p, s_1 and s_2, which are all given above, we get finally

$$s = \frac{2d^2n^3 + 3d(a+b-d)n^2 + \{d^2-3d(a+b)+6ab\}n}{1.2.3} \quad \ldots (6),$$

which is an expression* for the sum of n terms of the series

$$ab+(a+d)(b+d)+(a+2d)(b+2d) + \ldots + \{a+(n-1)d\}\{b+(n-1)d\}.$$

If $b = a$, then the series is

$$a^2+(a+d)^2 + (a+2d)^2 + \ldots\ldots + \{a+(n-1)d\}^2,$$

and by (6) the sum of n terms of this series is

$$s = \frac{2d^2n^3 + 3d(2a-d)n^2 + (d^2-6ad+6a^2)n}{1.2.3} \quad \ldots\ldots (7).$$

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* This theorem, remarkable for its elegance and utility, is due to Mr Rutherford, and is believed to be new.

EXAMPLES.

1. Find the sum of n terms of the series

$$3 . 7 + 6 . 10 + 9 . 13 + 12 . 16 + 15 . 19 + \ldots\ldots$$

Here $a = 3$, $b = 7$, $d = 3$; therefore by (6)

$$s = \frac{18n^3 + 63n^2 + 45n}{1 . 2 . 3} = \frac{6n^3 + 21n^2 + 15n}{2} = \frac{3n(n+1)(2n+5)}{2}.$$

2. Find the sum of n terms of the series

$$1 + 3 + 6 + 15 + \ldots\ldots$$

This series is $= \frac{1}{2}(1 . 2 + 2 . 3 + 3 . 4 + 4 . 5 + \ldots\ldots)$; therefore if $a = 1$, $b = 2$, and $d = 1$, we have by (6)

$$2s = \frac{2n^3 + 6n^2 + 4n}{1 . 2 . 3}, \text{ or } s = \frac{n^3 + 3n^2 + 2n}{1 . 2 . 3} = \frac{n(n+1)(n+2)}{1 . 2 . 3}.$$

3. Find the sum of the series

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots\ldots + n^2.$$

Here $a = b = d = 1$; therefore (6) or (7)

$$s = \frac{2n^3 + 3n^2 + n}{1 . 2 . 3} = \frac{n(n+1)(2n+1)}{1 . 2 . 3}.$$

4. Find the sum of n terms of the series

$$2 . 5 + 4 . 7 + 6 . 9 + 8 . 11 + \ldots\ldots$$

Ans. $\frac{n(n+1)(4n+11)}{3}$.

5. Find the sum of n terms of the series

$$2^2 + 5^2 + 8^2 + 11^2 + \ldots\ldots$$

Ans. $\frac{n(6n^2+3n-1)}{2}$.

GEOMETRICAL PROGRESSION.

A *geometrical progression* is a series of quantities in which the ratio of every two consecutive terms is the *same*. Thus

$$1, 2, 4, 8, 16, \ldots\ldots \text{ and } 27, 18, 12, 8, 5\tfrac{1}{3}, \ldots\ldots$$

are geometrical progressions, the former being an *increasing* series whose *common ratio* is 2, and the latter a *decreasing* one, whose common ratio is $\frac{2}{3}$. A geometrical progression is an increasing one when the common ratio is *greater* than unity, a decreasing one when it is *less* than unity, and the common ratio is found by dividing any term of the series by the term immediately preceding.

Let $a =$ the first term of a geometrical series,

$r =$ the common ratio,

$n =$ the number of terms,

$l =$ the last term, and

$s =$ the sum of n terms of the series, then we shall have

$$s = a + ar + ar^2 + ar^3 + ar^4 + \ldots\ldots \quad + ar^{n-2} + ar^{n-1},$$

where the index of r in any term is *one less* than the number of the term; hence we have

$$l = ar^{n-1} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

$$\text{Again, } s = a + ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-2} + ar^{n-1},$$

$$\therefore \quad rs = \quad ar + ar^2 + ar^3 + \ldots\ldots \quad + ar^{n-1} + ar^n.$$

Subtracting, $rs - s = ar^n - a$, or $s = a\dfrac{r^n - 1}{r - 1}$ . . . (2)

The equations (1) and (2) are sufficient for the solution of all questions which can be proposed in this branch of Algebra, and contain the entire *Theory* of Geometrical Progression.

When the ratio, r, is a proper fraction, the series is a descending or converging series, and admits of a *limit* to which it continually approximates, as the number of its terms is increased. But when r

is a proper fraction, r^n approximates to 0 as its limit; hence, in this case, the formula (2) becomes

$$s = \frac{a}{1 - r} \quad . \; . \; . \; . \; . \; . \quad (3)$$

A *geometrical mean* is the term, or *geometrical means* are the terms which lie between any two given terms or numbers.

To insert any number of geometrical means between two given quantities.

Let a and l be the two given quantities, and m the number of means to be inserted between them; then since there are m means and two extremes, we shall have $n = m + 2$, and by (1),

$$l = ar^{m+1}; \quad \therefore \quad r = \left(\frac{l}{a}\right)^{\frac{1}{m+1}} \quad . \; . \; . \quad (4)$$

This formula will determine the ratio, and the first quantity, a, multiplied by the ratio will give the first mean, and the first mean multiplied by the ratio will give the second mean, and so on, for all the means.

EXAMPLES.

1. Find the tenth term and the sum of ten terms of the series

$$1 - 2 + 2^2 - 2^3 + 2^4 - \ldots\ldots\ldots$$

Here $a = 1$, $r = \frac{-2}{1} = -2$, and $n = 10$; therefore

by (1), $l = ar^{n-1} = (-2)^9 = -512 =$ the tenth term;

and by (2), $s = a \cdot \frac{r^n - 1}{r - 1} = \frac{1024 - 1}{-2 - 1} = -341 =$ sum of 10 terms.

2. Insert three geometrical means between 4 and 64.

Here $a = 4$, $n = 5$, and $l = 64$; hence by (1)

$64 = 4r^4$, or $r^4 = 16$, $\therefore$ $r^2 = 4$, and $r = 2$, the ratio.

Hence the three means are 8, 16, and 32.

3. Find the sum of the converging series

$$100 + 40 + 16 + \ldots\ldots \textit{ in infinitum.}$$

Here $a = 100$, $r = \dfrac{40}{100} = \dfrac{2}{5}$, and therefore by (3)

$$s = \frac{a}{1-r} = \frac{100}{1-\frac{2}{5}} = \frac{500}{3} = 166\tfrac{2}{3}.$$

4. Find the value of the recurring decimal $\cdot 21\dot{6}\dot{5}$.

$$\text{Here } \cdot 21\dot{6}\dot{5} = \frac{21}{10^2} + \frac{65}{10^4} + \frac{65}{10^6} + \frac{65}{10^8} + \ldots\ldots \textit{ in infinitum}$$

$$= \frac{21}{10^2} + \frac{65}{10^4}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + \ldots\ldots \textit{ in inf.}\right)$$

Hence, taking $a = 1$, $r = \dfrac{1}{10^2} = \dfrac{1}{100}$, we have by (3)

$$1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\ldots \textit{ in inf.} = \frac{1}{1 - \frac{1}{100}} = \frac{100}{99};$$

$$\text{therefore } \cdot 21\dot{6}\dot{5} = \frac{21}{100} + \frac{65}{10000} \times \frac{100}{99} = \frac{21}{100} + \frac{13}{1980} = \frac{536}{2475}.$$

5. Sum the following series :—

(*a*) $1 + 2 + 4 + \ldots\ldots$ to 10 terms.	Ans. 1023.
(*b*) $1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots$ to 8 terms.	Ans. $1\frac{127}{128}$.
(*c*) $\frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \ldots\ldots$ to 5 terms.	Ans. $\frac{121}{162}$.
(*d*) $1 + \frac{1}{3} + \frac{1}{9} + \ldots\ldots$ to 10 terms.	Ans. $1\frac{9841}{19683}$.
(*e*) $3 + 4\frac{1}{2} + 6\frac{3}{4} + \ldots\ldots$ to 5 terms.	Ans. $39\frac{9}{16}$.
(*f*) $4 + 3 + \frac{9}{4} + \ldots\ldots$ *in infinitum.*	Ans. 16.
(*g*) $\frac{1}{2} - \frac{1}{3} + \frac{2}{9} - \ldots\ldots$ *in infinitum.*	Ans. $\frac{3}{10}$.

6. A person being asked to dispose of a horse, said he would sell him on condition of having a farthing for the first nail in his shoes, a halfpenny for the second, a penny for the third, two pence for the fourth, and so on, doubling the price of every nail, to 32, the number of nails in his four shoes; what would the horse be sold for at that rate?

Ans. L.4,473,924, 5s. 3¾d.

7. The second term of a geometrical series is 4, and the fifth term is 256; find the series. Ans. 1, 4, 16, 64, 256, *etc.*

8. The sum of 10 terms of a geometrical progression is $1133\frac{77}{256}$, and the ratio is $\frac{3}{2}$; what is the first term? Ans. 10.

9. The ratio being $-\frac{5}{3}$, the number of terms 5, and the sum of those terms $\frac{1684}{25}$; find the seventh term of the series. Ans. $\frac{2500}{9}$.

10. Find the value of the recurring decimals $2\cdot\dot{1}4285\dot{7}$ and $6\cdot5\dot{0}2\dot{3}$.
Ans. $\frac{15}{7}$, and $\frac{32479}{4995}$.

11. The difference of two numbers is 150, and the arithmetical mean exceeds the geometrical mean by one-sixth part of their difference; find the numbers. Ans. 50 and 200.

12. Find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84. Ans. 2, 4, and 8.

13. Find four numbers in geometrical progression, such that their sum shall be 15, and the sum of their squares 85.
Ans. 1, 2, 4, and 8.

14. The product of three numbers in geometrical progression is 64, and the sum of their cubes is 584; find them.
Ans. 2, 4, and 8.

15. If s_1, s_2, s_3 be the sums of three geometrical progressions, each continued *in infinitum*, whose first terms are a, b, c respectively, and whose common ratios are $\frac{1}{r}$, $-\frac{1}{r}$ and $\frac{1}{r^2}$; then will

$$\frac{s_1 s_2}{s_3} = \frac{ab}{c}.$$

16, If s_1, s_2, s_3 be the sums of n, $2n$, and $3n$ terms of a geometrical progression; then will

$$s^2_1 + s^2_2 = s_1 (s_2 + s_3).$$

GENERAL FORMULA

For determining the sum of n terms of a series of which the terms are the products of the corresponding terms of an arithmetical and a geometrical series.

Let $a + (a + d) + (a + 2d) + \ldots\ldots \{a + (n - 1)d\}$ be the arithmetical series, and

$s = 1 + r + r^2 + r^3 + \ldots\ldots r^{n-1}$, the geometrical series, omitting the common factor in each of the terms of the latter, as being readily supplied in the result; then we have to find the value of s, when

$$\text{s} = a + (a + d)r + (a + 2d)r^2 + \ldots\ldots \{a + (n-1)d\}r^{n-1}.$$

$$\text{Now } r\,\text{s} = \qquad ar + (a + d)r^2 + \ldots\ldots \{a + (n-2)d\}r^{n-1} + \{a + (n-1)d\}r^n;$$

$$\begin{aligned} \therefore\quad \text{s}(1-r) &= a + dr(1+r+r^2+ \ldots\ldots + r^{n-2}) - \{a+(n-1)d\}r^n; \\ &= a + dr(s - r^{n-1}) - \{a + (n-1)d\}r^n \\ &= a + drs - ar^n - dnr^n; \end{aligned}$$

$$\therefore\quad \text{s} = a\,.\frac{r^n - 1}{r-1} + \frac{dr(nr^{n-1} - s)}{r-1} = \frac{(a+nd)r^n - (a+drs)}{r-1} \ldots (1)$$

When r is a proper fraction, and the series continued *in infinitum*, then r^{n-1}, r^n, and nr^n are severally $= 0$; hence in this case

$$s = \frac{1}{1-r}, \text{ and } \text{s} = \frac{a}{1-r} + \frac{dr}{(1-r)^2} \ldots\ldots (2).$$

EXAMPLES.

1. Find the sum of n terms of the series

$$2.1 + 5.3 + 8.9 + 11.27 + \ldots\ldots$$

Here $a = 2$, $d = 3$, and $r = 3$; therefore $s = \frac{3^n - 1}{2}$, and by (1)

$$\text{s} = 2\,.\frac{3^n - 1}{3 - 1} + \frac{9(2n\,3^{n-1} - 3^n + 1}{2^2} = \frac{(6n-5)\,3^n + 5}{4}.$$

2. Find the sum of 9 terms of the series

$$1.1 - 3.3 + 5.9 - 7.27 + \ldots\ldots$$

Ans. 86113.

3. Find the sum of 6 terms of the series

$$1 + 1 + \tfrac{5}{9} + \tfrac{7}{27} + \tfrac{9}{81} + \ldots\ldots$$

and also its value *in infinitum*.

Ans. $\frac{722}{243}$ and 3.

HARMONICAL PROGRESSION.

An *harmonical progression* is a series of which the first of any *three* consecutive terms is to the third, as the difference between the first and second is to the difference between the second and third. Thus,

$$1,\ \tfrac{1}{3},\ \tfrac{1}{5},\ \tfrac{1}{7},\ \tfrac{1}{9},\ \ldots\ldots \text{ and } 2,\ 2\tfrac{2}{5},\ 3,\ 4,\ 6,\ 12,\ \ldots\ldots$$

are harmonical series, for taking any three consecutive terms of either of these series, the preceding condition expressed in the definition is fulfilled.

The reciprocals of quantities in harmonical progression are in arithmetical progression.

Let a, b, c, d be four quantities in harmonical progression; then by the definition

$$a : c :: a - b : b - c, \text{ and } b : d :: b - c : c - d,$$

$$\text{or } ab - ac = ac - bc, \text{ and } bc - bd = bd - cd;$$

$$\therefore \frac{ab - ac}{abc} = \frac{ac - bc}{abc}, \text{ and } \frac{bc - bd}{bcd} = \frac{bd - cd}{bcd},$$

$$\text{or } \frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}, \text{ and } \frac{1}{d} - \frac{1}{c} = \frac{1}{c} - \frac{1}{b},$$

$$\therefore \frac{1}{a} - \frac{1}{b} = \frac{1}{b} - \frac{1}{c} = \frac{1}{c} - \frac{1}{d};$$

hence $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, and $\frac{1}{d}$ are in arithmetical progression.

Since an arithmetical series may have a term $=0$, therefore an harmonical series may have a term $= \frac{1}{0} = \infty$.

Thus, if three terms of an harmonical series are 2, 3, 6, and if it be required to continue the series both ways to three terms; then inverting the terms of the harmonical series, we have $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$ in arithmetical progression, the common difference being $\frac{1}{6}$; hence continuing the arithmetical series both ways, we get

$$\ldots\ldots 1,\ \tfrac{5}{6},\ \tfrac{2}{3},\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{6},\ 0,\ -\tfrac{1}{6},\ -\tfrac{1}{3},\ \ldots\ldots$$

and the harmonical series is therefore,

$$\ldots\ldots 1,\ \tfrac{6}{5},\ \tfrac{3}{2},\ 2,\ 3,\ 6,\ \infty,\ -6,\ -3,\ \ldots\ldots$$

In consequence of this connexion between the terms of an arithmetical and an harmonical progression, all questions that can be proposed in the latter may be solved by the principles of the former, and we shall only hint at a few of them, as their solution can present no difficulty.

EXAMPLES.

1. Insert 6 harmonical means between 3 and $\frac{6}{23}$.
Ans. $\frac{6}{5}$, $\frac{3}{4}$, $\frac{6}{11}$, $\frac{3}{7}$, $\frac{6}{17}$, and $\frac{3}{10}$.

2. Find a fourth harmonical proportional to 6, 8 and 12.
Ans. 24.

3. The arithmetical mean between two numbers exceeds the geometrical by 13, and the geometrical exceeds the harmonical by 12, what are the numbers? Ans. 104 and 234.

4. If m be the first of three harmonical means inserted between a and b; shew that $a : b :: 3m : 4b - m$.

5. There are four quantities, of which the first three are in arithmetical progression, and the last three in harmonical progression, prove that the four quantities are proportionals.

6. If a, b, c be respectively the pth, qth, and rth terms of an harmonical series, shew that

$$\frac{p-q}{c} + \frac{r-p}{b} + \frac{q-r}{a} = 0.$$

PILING OF SHOT AND SHELLS.

Shot and shells are usually piled in three different forms, and the pile is termed *triangular, square*, or *rectangular*, according as its base is triangular, square, or rectangular.

Let n denote the number of shot in either of the outer rows of the base, or bottom course of the triangular pile; then $n - 1$ will be the number in the second row of the base, or in the outer row of the second course; $n - 2$, the number in the third row of the base, or in the outer row of the third course, and so on, till the last row of the base, or the top course, which consists of a single shot; hence the number of courses in the pile will be equal to the number of shot in the outer row of the base.

Now the number of shot in the bottom course will be the sum of the series

$$1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2} \; \ldots\ldots \; (1).$$

Diminishing n successively by unity in this formula, we should obtain the number of shot in each course, reckoning upwards from the bottom row; hence reversing this process, and expounding n successively by 1, 2, 3, 4, ... n, we get the number of shot in the several courses, commencing at the top course. Therefore the number of shot in the *triangular pile* will be the sum of the series

$$1 + 3 + 6 + 10 + 15 + \ldots\ldots + \frac{n(n+1)}{2}.$$

In a similar manner it is evident that the number of courses in the square pile is equal to the number of shot in the outer row of the base; hence the number of shot in the *square pile* will be the sum of the series

$$1 + 4 + 9 + 16 + 25 + \ldots\ldots + n^2.$$

The number of courses in a rectangular pile, which terminates in a single row of shot at top, is equal to the number of shot in the breadth of the rectangular base; and if the longer side of the base contain $m + n$ shot, the bottom course will contain $n(m + n)$ shot. Diminishing n successively by unity, we should have the number of shot in the several courses reckoning upwards, and when $n = 1$, then $m+n$ will become $m+1$, which is the number of shot in the top row of the pile. Hence reversing this process, and expounding n successively by 1, 2, 3, 4, ... n, we get the number of shot in the *rectangular pile* equal to the sum of the series

$$1(m+1)+2(m+2)+3(m+3) + \ldots\ldots +n(m+n).$$

The summation of this class of series has been already obtained in the *general formula* (6) given at p. 159, and instead of repeating the investigation here, we shall deduce the requisite formulæ from the general one referred to in the Arithmetical Progression. The general formula is

$$s = ab+(a+d)(b+d)+(a+2d)(b+2d)+\ldots\{a+(n-1)d\}\{b+(n-1)d\}$$

$$= \frac{2dn^3 + 3d(a + b - d)n^2 + \{d^2 - 3d(a + b) + 6ab\}n}{1 \,.\, 2 \,.\, 3}.$$

1. *To find the number of shot in a Triangular Pile.*

Let $a = 1$, $b = 2$, and $d = 1$; then by the preceding formula,

$$1 \,.\, 2 + 2 \,.\, 3 + 3 \,.\, 4 + \ldots\ldots + n(n + 1) = \frac{2n^3 + 6n^2 + 4n}{1 \,.\, 2 \,.\, 3},$$

and dividing each term by 2 gives

$$1+3+6+10+\ldots\ldots+\frac{n(n+1)}{2} = \frac{n^3+3n^2+2n}{1 \,.\, 2 \,.\, 3} = \frac{n(n+1)(n+2)}{1 \,.\, 2 \,.\, 3}.$$

2. *To find the number of shot in a Square Pile.*

Let $a = 1$, $b = 1$, $d = 1$; then we have

$$1^2+2^2+3^2+4^2+\ldots\ldots+n^2 = \frac{2n^3+3n^2+n}{1 \,.\, 2 \,.\, 3} = \frac{n(n+1)(2n+1)}{1 \,.\, 2 \,.\, 3}.$$

3. *To find the number of shot in a Rectangular Pile.*

Let $a = 1$, $b = m + 1$, and $d = 1$; then we have

$$1(m+1)+2(m+2)+3(m+3)+\ldots n(m+n) = \frac{2n^3+3(m+1)n^2+(3m+1)n}{1 \,.\, 2 \,.\, 3}$$

$$= \frac{2n^3 + 3n^2 + n + 3mn(n + 1)}{1 \,.\, 2 \,.\, 3} = \frac{n(n + 1)(2n + 1 + 3m)}{1 \,.\, 2 \,.\, 3}.$$

These three formulæ for the number of shot in the three different piles have a common factor, viz., $\frac{n(n+1)}{2}$, which expresses the number of shot in a triangular face of the pile, and since

$$\frac{n(n+1)(n+2)}{2\,.\,3} = \frac{n(n+1)}{2}\,.\,\frac{n+1+1}{3},$$

$$\frac{n(n+1)(2n+1)}{2\,.\,3} = \frac{n(n+1)}{2}\,.\,\frac{n+1+n}{3},$$

$$\frac{n(n+1)(2n+1+3m)}{2\,.\,3} = \frac{n(n+1)}{2}\,.\,\frac{(n+m)+(m+1)+(n+m)}{3},$$

we may deduce the following rule which applies to all the piles:

Multiply the number of shot in a triangular face of the pile, by one third the sum of the number of shot in the three parallel edges, and the product will be the number of shot in the pile.

In the triangular pile, the shot at top, one edge of the base, and the shot at the vertex of the triangular base, opposite to the edge, are considered the three parallel edges. In the square pile, the single shot at the top is one of the parallel edges.

The number of shot in an *incomplete pile* is evidently the difference between the number of shot in the complete pile, and the number in the pile wanting.

EXAMPLES.

1. Find the number of shot in a triangular pile of 20 courses.

Here $\frac{n(n+1)(n+2)}{2\,.\,3} = \frac{20 \times 21 \times 22}{2\,.\,3} = 10 \times 7 \times 22 = 1540.$

2. Required the number of shot in an incomplete rectangular pile of 7 courses, having 10 and 20 shot respectively in the shorter and longer sides of the upper course.

The number of shot in the longer and shorter sides of the pile wanting are 19 and 9, and the number in the longer and shorter sides of the bottom course are 26 and 16; hence, for the whole pile, $m + n = 26$ and $n = 16$; therefore $m = 10$, and

$$\frac{n(n+1)(2n+1+3m)}{2\,.\,3} = \frac{16\times17\times63}{2\,.\,3} = 8\times17\times21 = 2856.$$

Again, for the pile wanting, we have $m + n = 19$ and $n = 9$; hence $m = 10$ as before, and

$$\frac{n(n+1)(2n+1+3m)}{2\,.\,3} = \frac{9\times10\times49}{2\,.\,3} = 3\times5\times49 = 735;$$

therefore the number of shot in the incomplete pile is $= 2856 - 735 = 2121$.

3. Find the number of shot in a square pile of 14 courses.
Ans. 1015.

4. Find the number of shot in an incomplete triangular pile of 17 courses, having 15 shot in the top course. Ans. 1751

5. The number of shot in the base and top courses of a square pile are 1521 and 169; how many are in the incomplete pile? Ans. 19890.

6. How many shot are required to complete a rectangular pile, when the difference between the number of shot in the two sides of its base is 7, and the number of shot in the longer side of its upper course is 15?
Ans. 336.

7. The longer side of the base of a rectangular pile is double the shorter, and sixteen times the number of shot in a triangular pile, together with eleven times the number in a square pile are equal to five times the number in the rectangular pile, each having the same number of courses; find the number of courses in each pile. Ans. 19.

8. If p denote the number of courses in an incomplete rectangular pile, and $3h$ and h the number of shot in the longer and shorter sides of the upper course, shew that the number of shot is

$$\frac{p\{2(p+3h)^2 - 3(p+4h) + 1\}}{1\,.\,2\,.\,3}.$$

9. An incomplete rectangular pile has 13 courses left: the number of shot in the longer side of the base course is to the number in its shorter side as 7 to 5, and in the top course as 3 to 1 find the number of shot in the original complete pile. Ans. 1960.

INDETERMINATE COEFFICIENTS.

The principle of the method of indeterminate coefficients consists in assuming a series of terms with unknown coefficients for the required quantity, and will be easily understood from the following

THEOREM.

If the equation

$$a+bx+cx^2+dx^3+\ldots\ldots=a'+b'x+c'x^2+d'x^3+\ldots\ldots$$

subsist, whatever be the value of x; then the coefficients of the powers of x in the one member, will be equal to the coefficients of the same powers of x in the other member; that is,

$$a=a',\quad b=b',\quad c=c',\quad d=d', \text{ etc.}$$

For, by transposition, we have

$$a-a'+(b-b')x+(c-c')x^2+(d-d')x^3+\ \ldots\ldots=0\ \ldots\ldots (1);$$

and if the coefficients of the several powers are not simultaneously equal to 0, but have finite values, then the equation (1) can only be satisfied by those values of x which are *roots* of the equation, and not by *all* values of x; hence equation (1) cannot subsist universally, unless $a-a'=0$, $b-b'=0$, $c-c'=0$, *etc.*, and hence we must have

$$a=a',\quad b=b',\quad c=c',\quad d=d', \text{ etc.}$$

EXAMPLES.

1. Expand $\dfrac{a}{1+x+2x^2}$ in ascending powers of x, by the method of indeterminate coefficients.

Assume $\dfrac{a}{1+x+2x^2}=A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\ldots\ldots,$

then multiplying the second member by $1+x+2x^2$ we get

$$\begin{array}{rl|l|l|l|l} a = & \text{A}_0 + \text{A}_1 & x + \text{A}_2 & x^2 + \text{A}_3 & x^3 + \text{A}_4 & x^4 + \ldots . \\ & + \text{A}_0 & + \text{A}_1 & + \text{A}_2 & + \text{A}_3 & \\ & & +2\text{A}_0 & +2\text{A}_1 & +2\text{A}_2 & \end{array}$$

Hence by the preceding theorem, we get

$$\text{A}_0 = a,\ \text{A}_1 + \text{A}_0 = 0,\ \text{A}_2 + \text{A}_1 + 2\text{A}_0 = 0,\ \text{A}_3 + \text{A}_2 + 2\text{A}_1 = 0, \textit{ etc.};$$

$$\therefore\ \text{A}_1 = -\text{A}_0 = -a,\ \ \text{A}_2 = -\text{A}_1 - 2\text{A}_0 = -a,\ \ \text{A}_3 = -\text{A}_2 - 2\text{A}_1 = 3a, \textit{ etc.}$$

Carrying on this process a little further, and substituting for A_0, A_1, A_2, A_3, *etc.*, their values thus found, we get

$$\frac{a}{1 + x + 2x^2} = a - ax - ax^2 + 3ax^3 - ax^4 - 5ax^5 + 7ax^6 + \ldots . .$$

2. Resolve $\dfrac{1 - x - 4x^2}{x - 4x^2 + 3x^3}$ into partial fractions.

Here one factor of the denominator is x, and putting the other factor $1 - 4x + 3x^2 = 0$, and resolving the quadratic, we get $x = 1$ or $x = \frac{1}{3}$; hence $1 - x$ and $1 - 3x$ are the factors of $1 - 4x + 3x^2$, and $x - 4x^2 + 3x^3 = x(1 - x)(1 - 3x)$. Now assume

$$\frac{1 - x - 4x^2}{x - 4x^2 + 3x^3} = \frac{\text{A}}{x} + \frac{\text{B}}{1 - x} + \frac{\text{C}}{1 - 3x};$$

then reducing the partial fractions in the second member to a common denominator, viz., $x - 4x^2 + 3x^3$, and equating the numerators of both members, we get

$$1 - x - 4x^2 = \text{A} - (4\text{A} - \text{B} - \text{C})x + (3\text{A} - 3\text{B} - \text{C})x^2.$$

Comparing the coefficients of the same powers of x in both members, we have

$$\text{A} = 1,\quad 4\text{A} - \text{B} - \text{C} = 1,\quad 3\text{A} - 3\text{B} - \text{C} = -4;$$

and from these, since $\text{A} = 1$, we derive $3\text{B} + \text{C} = 7$, and $\text{B} + \text{C} = 3$; hence $\text{B} = 2$ and $\text{C} = 1$; consequently

$$\frac{1 - x - 4x^2}{x - 4x^2 + 3x^3} = \frac{\text{A}}{x} + \frac{\text{B}}{1 - x} + \frac{\text{C}}{1 - 3x} = \frac{1}{x} + \frac{2}{1 - x} + \frac{1}{1 - 3x}.$$

3. Expand $\dfrac{1 + 2x}{1 - x - x^2}$ in a series.

Ans. $1 + 3x + 4x^2 + 7x^3 + 11x^4 + 18x^5 + \ldots .$

4. Let it be required to develop $\frac{3+2x}{5+4x}$ in a series.

$$\text{Ans. } \frac{3}{5} - \frac{2}{5^2}x + \frac{2^3}{5^3}x^2 - \frac{2^5}{5^4}x^3 + \frac{2^7}{5^5}x^4 - \ldots\ldots$$

5. Let it be required to expand $\sqrt{(a^2-x^2)}$ in a series.

$$\text{Ans. } a - \frac{1}{2}\cdot\frac{x^2}{a} - \frac{1}{2.4}\cdot\frac{x^4}{a^3} - \frac{1.3}{2.4.6}\cdot\frac{x^6}{a^5} - \ldots\ldots$$

6. Resolve $\frac{6x^2-4x-6}{(x-1)(x-2)(x-3)}$ and $\frac{x^2}{(x^2-1)(x-2)}$ into partial fractions.

$$\text{Ans. } \frac{18}{x-3} - \frac{10}{x-2} - \frac{2}{x-1},$$

$$\text{and } \frac{1}{6(x+1)} + \frac{4}{3(x-2)} - \frac{1}{2(x-1)}.$$

7. If $x = y - y^2$, find the value of y in terms of x.

$$\text{Ans. } y = 1 - x - x^2 - 2x^3 - 5x^4 - 14x^5 - \ldots\ldots$$
$$\text{or } y = x + x^2 + 2x^3 + 5x^4 + 14x^5 + \ldots\ldots$$

BINOMIAL THEOREM.

The expanded value of $(a+b)^n$, when n is a positive integer number, has been given at p. 49, and it is now proposed to shew that the form of the expansion of $(a+x)^n$ remains the same, whether n be positive or negative, integral or fractional.

Since $(a+x)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n = a^n(1+z)^n$, if $z = \frac{x}{a}$, it will be sufficient to confine the demonstration to the simpler form $(1+z)^n$, and thence deduce the expansion of the more general form $(a+x)^n$.

First, let n be a positive integer; then, by actual multiplication, we know that $(1+z)^2 = 1 + 2z + z^2$, $(1+z)^3 = 1 + 3z + \ldots\ldots$, and so on, where the coefficient of z in the second member is the index of the power. Now assume generally that

$$(1+z)^{n-1} = 1 + (n-1)z + \dots\dots,$$

and multiply both members by $1+z$; then, we have

$$(1+z)^n = 1 + nz + \dots\dots;$$

Hence if this law holds true for any one index, it is also true for the next greater index; but it is true for $n=2$, $n=3$; hence it is true for $n=4$, $n=5$, and so on, and therefore it is universally true.

Second, let n be a positive fraction, or $n = \frac{p}{q}$; and assume that

$$(1+z)^{\frac{p}{q}} = 1 + \text{A}z + \dots\dots\dots,$$

and raising both members to the qth power, we get

$$(1+z)^p = (1 + \text{A}z + \dots\dots)^q,$$

or $1 + pz + \dots = 1 + q\text{A}z + \dots\dots$, since p and q are positive integers; hence, equating the coefficients of the first power of z in both members, we get $p = q\text{A}$, or $\text{A} = \frac{p}{q}$; and therefore in this case also

$$(1+z)^{\frac{p}{q}} = 1 + \frac{p}{q}z + \dots\dots$$

Lastly, let n be negative, and integral or fractional; then if $n = -r$,

$$(1+z)^n = (1+z)^{-r} = \frac{1}{(1+z)^r} = \frac{1}{1 + rz + \dots\dots}$$

$$= 1 - rz + \dots\dots,$$

by actual division; and therefore in all cases the form of the expansion is

$$(1+z)^n = 1 + nz + \text{B}z^2 + \text{C}z^3 + \text{D}z^4 + \dots\dots \quad (1),$$

and it only remains to determine the values of B, C, D, *etc.* in terms of n.

Let $z = y + v$; then substituting in (1) $y + v$ for z, we get

$$
\begin{aligned}
(1 + y + v)^n &= 1 + n(y + v) + \mathrm{B}(y + v)^2 + \mathrm{C}(y + v)^3 + \ldots . \\
&= 1 + ny + \mathrm{B}y^2 + \mathrm{C}y^3 + \mathrm{D}y^4 + \ldots . \\
&\quad + (n + 2\mathrm{B}y + 3\mathrm{C}y^2 + 4\mathrm{D}y^3 + \ldots)v \\
&\quad + (\mathrm{B} + 3\mathrm{C}y + \ldots .)v^2 \\
&\quad \vdots
\end{aligned}
$$

$$\text{or } (1+y+v)^n = (1+y)^n + (n+2\mathrm{B}y+3\mathrm{C}y^2+4\mathrm{D}y^3+ \ldots)v + \ldots \quad (2).$$

$$
\begin{aligned}
\text{But } (1+y+v)^n &= \{(1 + y) + v\}^n = (1+y)^n\left\{1 + \frac{v}{1+y}\right\}^n \\
&= (1+y)^n\left\{1+n\frac{v}{1+y} + \mathrm{B}\frac{v^2}{(1+y)^2} + \mathrm{C}\frac{v^3}{(1+y)^3} + \ldots\right\} \\
&= (1+y)^n + n(1 + y)^{n-1}.v + \mathrm{B}(1 + y)^{n-2}.v^2 + \ldots . \quad (3).
\end{aligned}
$$

But these two developments of $(1 + y + v)^n$ must be identical; hence equating the coefficients of v in (2) and (3), we get

$$n + 2\mathrm{B}y + 3\mathrm{C}y^2 + 4\mathrm{D}y^3 + \ldots . = n(1+y)^{n-1},$$

and multiplying both sides by $1 + y$, gives

$$
\begin{array}{r|r|r|l}
n + 2\mathrm{B} & y + 3\mathrm{C} & y^2 + 4\mathrm{D} & y^3 + \ldots . = n(1 + y)^n \\
+\ n & +\ 2\mathrm{B} & +\ 3\mathrm{C} & \qquad\quad = n + n^2y + \mathrm{B}ny^2 + \mathrm{C}ny^3 + \ldots
\end{array}
$$

Equating the coefficients of the same powers of y, we obtain

$$
\begin{aligned}
2\mathrm{B} + n &= n^2, & \therefore\ \mathrm{B} &= \frac{n^2 - n}{2} = \frac{n(n-1)}{2}, \\
3\mathrm{C} + 2\mathrm{B} &= \mathrm{B}n, & \therefore\ \mathrm{C} &= \frac{\mathrm{B}(n-2)}{3} = \frac{n(n-1)\ (n-2)}{2.3}, \\
4\mathrm{D} + 3\mathrm{C} &= \mathrm{C}n, & \therefore\ \mathrm{D} &= \frac{\mathrm{C}(n-3)}{4} = \frac{n(n-1)\ (n-2)\ (n-3)}{2.3.4}, \\
&\vdots & &\vdots \qquad\qquad \vdots
\end{aligned}
$$

$$\text{Hence } (1 + z)^n = 1 + nz + \frac{n(n-1)}{1.\ 2}z^2 + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}z^3 + \ldots,$$

and writing $\frac{x}{a}$ for z, and multiplying by a^n, we get finally

$$(a + x)^n = a^n\left(1 + \frac{x}{a}\right)^n = a^n\left\{1+n\frac{x}{a} + \frac{n(n-1)}{1.\ 2}.\frac{x^2}{a^2} + \ldots . .\right\}$$

or

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}a^{n-3}x^3 +$$

$$\ldots\ldots\ldots + \frac{n(n-1)(n-2)\ldots\ldots\{n-(p-1)\}}{1.\ 2.\ 3\ldots\ldots p}a^{n-p}x^p;$$

where the last term is the $(p+1)$th term of the expansion. Writing $-x$ for x, we get

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 - \frac{n(n-1)(n-2)}{1.2.3.}a^{n-3}x^3 + \ldots.$$

$$\pm\frac{n(n-1)(n-2)\ldots\ldots\{n-(p-1)\}}{1.2.3\ldots\ldots p}a^{n-p}x^p;$$

the $(p+1)$th term being $+$ or $-$, according as p is an even or an odd number.

Since

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1.\ 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}a^{n-3}x^3 + \ldots,$$

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1.\ 2}a^{n-2}x^2 - \frac{n(n-1)(n-2)}{1.\ 2.\ 3}a^{n-3}x^3 + \ldots.;$$

therefore by addition and subtraction, and dividing by 2, we get

$$\frac{(a+x)^n + (a-x)^n}{2} = \text{sum of all the odd terms of the expansion,}$$

$$\frac{(a+x)^n - (a-x)^n}{2} = \text{sum of all the even terms of the expansion.}$$

EXAMPLES.

1. Let it be required to expand $(a^2+x^2)^{\frac{3}{2}}$ in a series.

Comparing this with the form $(a+x)^n$, we have $a = a^2$, $x = x^2$, and $n = \frac{3}{2}$; therefore

$$(a^2+x^2)^{\frac{3}{2}} = (a^2)^{\frac{3}{2}} + \tfrac{3}{2}(a^2)^{\frac{1}{2}}x^2 + \frac{\frac{3}{2}\cdot\frac{1}{2}}{1\cdot 2}(a^2)^{-\frac{1}{2}}x^4 + \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot-\frac{1}{2}}{1\cdot\ 2\cdot\ 3}(a^2)^{-\frac{3}{2}}x^6 + etc.$$

$$= a^3 + \frac{3}{2}ax^2 + \frac{3\cdot 1}{2^2\cdot 1\cdot 2}\cdot\frac{x^4}{a} - \frac{3\cdot 1\cdot 1}{2^3\cdot 1\cdot 2\cdot 3}\cdot\frac{x^6}{a^3} + \frac{3\cdot 1\cdot 1\cdot 3}{2^4\cdot 1\cdot 2\cdot 3\cdot 4}\cdot\frac{x^8}{a^5} - \ldots\ldots$$

2. Expand $\dfrac{a^2}{(a^2-x)^{\frac{1}{2}}}$ in a series.

Here $\dfrac{a^2}{(a^2-x)^{\frac{1}{2}}} = a^2(a^2-x)^{-\frac{1}{2}}$, and expanding* the factor $(a^2-x)^{-\frac{1}{2}}$, we have $a = a^2$, $x = -x$ and $n = -\frac{1}{2}$; therefore

$$(a^2-x)^{-\frac{1}{2}} = (a^2)^{-\frac{1}{2}} - \tfrac{1}{2}(a^2)^{-\frac{3}{2}}.(-x) + \frac{-\frac{1}{2}\,.\,-\frac{3}{2}}{1\,.\,2}(a^2)^{-\frac{5}{2}}(-x)^2$$

$$+ \frac{-\frac{1}{2}\,.\,-\frac{3}{2}\,.\,-\frac{5}{2}}{1\,.\,2\,.\,3}(a^2)^{-\frac{7}{2}}(-x)^3 + \ldots$$

$$= a^{-1} + \frac{1}{2}a^{-3}x + \frac{1\,.\,3}{2^2\,.\,1\,.\,2}a^{-5}x^2 + \frac{1\,.\,3\,.\,5}{2^3.1\,.\,2\,.\,3}a^{-7}x^3 + \ldots;$$

$$\therefore \frac{a^2}{(a^2-x)^{\frac{1}{2}}} = a^2(a^2-x)^{-\frac{1}{2}} = a + \frac{1}{2}.\frac{x}{a} + \frac{1.3}{2^2.1.2}.\frac{x^2}{a^3} + \frac{1.3.5}{2^3.1.2.3}.\frac{x^3}{a^5} + \ldots$$

3. Expand $\left(\dfrac{a}{a-x}\right)^3$ in a series.

Here $\left(\dfrac{a}{a-x}\right)^3 = \dfrac{a^3}{(a-x)^3} = a^3(a-x)^{-3}$, and by the formula for the expansion of $(a-x)^n$ we have, when $n = -3$,

$$(a-x)^{-3} = a^{-3} - (-3)a^{-}x + \frac{-3\,.\,-4}{1\,.\,2}a^{-5}x^2$$

$$- \frac{-3\,.\,-4\,.\,-5}{1\,.\;\;2\,.\;\;3}a^{-6}x^3 + \ldots\ldots;$$

$$\therefore \left(\frac{a}{a-x}\right)^3 = a^3(a-x)^{-3} = a^0 + 3a^{-1}x + \frac{3\,.\,4}{1\,.\,2}a^{-2}x^2 + \frac{3\,.\,4\,.\,5}{1\,.\,2\,.\,3}a^{-3}x^3 + \ldots$$

$$= 1 + 3\frac{x}{a} + \frac{3\,.\,4}{1\,.\,2}.\frac{x^2}{a^2} + \frac{3\,.\,4\,.\,5}{1\,.\,2\,.\,3}.\frac{x^3}{a^3} + \frac{3\,.\,4\,.\,5\,.\,6}{1\,.\,2\,.\,3\,.\,4}.\frac{x^4}{a^4} + \ldots.$$

---

* The student should be familiar with the general form of the expansion of a binomial quantity, and the expansion of particular examples is nothing more than substitution in the general formula, and then simplifying the result of that substitution.

4. Find the fifth term of the expansion of $\dfrac{1}{(a^2-x^3)^{\frac{3}{4}}}$.

Here $\dfrac{1}{(a^2-x^3)^{\frac{3}{4}}}=(a^2-x^3)^{-\frac{3}{4}}$, and by comparison with the form $(a+x)^n$ we have $a=a^2$, $x=-x^3$ and $n=-\frac{3}{4}$; therefore when $p=4$, the $(p+1)$th or 5th term of the expansion is

$$\frac{-\frac{3}{4}\,.\,-\frac{7}{4}\,.\,-\frac{11}{4}\,.\,-\frac{15}{4}}{1\,.\,2\,.\,3\,.\,4}(a^2)^{-\frac{19}{4}}(-x^3)^4=\frac{3\,.\,7\,.\,11\,.\,15}{4^4\,.\,1\,.\,2\,.\,3\,.\,4}\,.\,a^{-\frac{19}{2}}x^{12}$$

$$=\frac{1155}{2048}\,.\,\frac{x^{12}}{a^{\frac{19}{2}}}.$$

EXAMPLES FOR PRACTICE.

Expand the following quantities by the binomial theorem.

	Answers.
1. $(a+x)^{\frac{1}{2}}$.	$a^{\frac{1}{2}}\left\{1+\frac{1}{2}\cdot\frac{x}{a}-\frac{1}{2\,.\,4}\cdot\frac{x^2}{a^2}+\frac{1\,.\,3}{2\,.\,4\,.\,6}\cdot\frac{x^3}{a^3}-\ldots\right\}$
2. $(a-x)^{\frac{1}{2}}$.	$a^{\frac{1}{2}}\left\{1-\frac{1}{2}\cdot\frac{x}{a}-\frac{1}{2\,.\,4}\cdot\frac{x^2}{a^2}-\frac{1\,.\,3}{2\,.\,4\,.\,6}\cdot\frac{x^3}{a^3}-\ldots\right\}$
3. $(a+b)^{\frac{1}{3}}$.	$a^{\frac{1}{3}}\left\{1+\frac{1}{3}\cdot\frac{b}{a}-\frac{2}{3\,.\,6}\cdot\frac{b^2}{a^2}+\frac{2\,.\,5}{3\,.\,6\,.\,9}\cdot\frac{b^3}{a^3}-\ldots\right\}$
4. $(a-b)^{\frac{1}{4}}$.	$a^{\frac{1}{4}}\left\{1-\frac{1}{4}\cdot\frac{b}{a}-\frac{3}{4\,.\,8}\cdot\frac{b^2}{a^2}-\frac{3\,.\,7}{4\,.\,8\,.\,12}\cdot\frac{b^3}{a^3}-\ldots\right\}$
5. $(a+x)^{\frac{2}{3}}$.	$a^{\frac{2}{3}}\left\{1+\frac{2}{3}\cdot\frac{x}{a}-\frac{1}{9}\cdot\frac{x^2}{a^2}+\frac{1\,.\,4}{9\,.\,9}\cdot\frac{x^3}{a^3}-\ldots\right\}$
6. $(1-x)^{\frac{2}{5}}$.	$1-\frac{2}{5}x-\frac{2\,.\,3}{5\,.\,10}x^2-\frac{2\,.\,3\,.\,8}{5\,.\,10\,.\,15}x^3-\ldots$
7. $\dfrac{1}{(1+x)^{\frac{1}{5}}}$.	$1-\frac{1}{5}x+\frac{6}{5\,.\,10}x^2-\frac{6\,.\,11}{5\,.\,10\,.\,15}x^3+\ldots$

8. $\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$. Ans. $1+\frac{x}{a}+\frac{1}{2}\cdot\frac{x^2}{a^2}+\frac{1}{2}\cdot\frac{x^3}{a^3}+\frac{3}{8}\cdot\frac{x^4}{a^4}+\ldots\ldots$

9. Find the fifth term of the expansion of $(a^{\frac{2}{3}} - x^{\frac{3}{2}})^{\frac{4}{5}}$.

Ans. $-\frac{11}{625}a^{-\frac{32}{15}}x^6$.

10. Find the fifth term of the expansion of $(ax - by)^{-10}$.

Ans. $715\cdot\frac{b^4y^4}{a^{14}x^{14}}$.

11. Shew that four times the product of the sums of the odd and even terms of the expansion of $(a + x)^n$ is $= (a + x)^{2n} - (a - x)^{2n}$.

12. Find the sum of the squares of the coefficients of $(1 + x)^n$.

Ans. $\frac{1\,.\,3\,.\,5\ldots\ldots(2n-1)}{1\,.\,2\,.\,3\ldots\ldots n}.2n$.

MULTINOMIAL THEOREM.

The *multinomial theorem* is a formula which exhibits the expansion, in a series ascending according to the powers of x, of

$$(a + bx + cx^2 + dx^3 + ex^4 + \ldots\ldots)^n.$$

Let $(a+bx+cx^2+dx^3+\ldots\ldots)^n = t_1+t_2x+t_3x^2+t_4x^3+\ldots,\ \ldots(1)$

$$(a+by+cy^2+dy^3+\ldots\ldots)^n = t_1+t_2y+t_3y^2+t_4y^3+\ldots,$$

$$a+bx+cx^2+dx^3+\ldots\ldots = u, \text{ and } a+by+cy^2+dy^3+\ldots = v;$$

$$\text{then } \frac{u^n-v^n}{u-v} = \frac{t_2(x-y)+t_3(x^2-y^2)+t_4(x^3-y^3)+\ldots}{b(x-y)+c(x^2-y^2)+d(x^3-y^3)+\ldots}$$

$$= \frac{t_2+t_3(x+y)+t_4(x^2+xy+y^2)+\ldots}{b+c(x+y)+d(x^2+xy+y^2)+\ldots}.$$

But $u^n - v^n$ divided by $u - v$ gives a quotient, $u^{n-1} + u^{n-2}v + u^{n-3}v^2 + \ldots + uv^{n-2} + v^{n-1}$, consisting of n terms, which when

$u = v$ becomes nu^{n-1}; hence, if in the last equation we make $x = y$, then also $u = v$, and it reduces to

$$nu^{n-1} \text{ or } \frac{nu^n}{u} = \frac{t_2+2t_3x+3t_4x^2+4t_5x^3+\dots}{b+2cx+3dx^2+4ex^3+\dots};$$

$$\therefore \frac{n(t_1+t_2x+t_3x^2+t_4x^3+\dots)}{a+bx+cx^2+dx^3+\dots} = \frac{t_2+2t_3x+3t_4x^2+4t_5x^3+\dots}{b+2cx+3dx^2+4ex^3+\dots}.$$

Clearing of fractions, multiplying out, and arranging the terms, we get

$$\begin{array}{lr|r|r|l}
at_2 & +\ 2at_3 & x\ +\ 3at_4 & x^2\ +\ 4at_5 & x^3+\dots \\
 & +\ bt_2 & +\ 2bt_3 & +\ 3bt_4 & \\
 & & +\ ct_2 & +\ 2ct_3 & \\
 & & & +\ dt_2 &
\end{array}$$

$$\begin{array}{lr|r|r|l}
=nbt_1 & +\ nbt_2 & x\ +\ nbt_3 & x^2\ +\ nbt_4 & x^3+\dots \\
 & +\ 2nct_1 & +\ 2nct_2 & +\ 2nct_3 & \\
 & & +\ 3ndt_1 & +\ 3ndt_2 & \\
 & & & +\ 4net_1 &
\end{array}$$

Equating the coefficients of the same powers of x in both members, gives

$$at_2 = nbt_1$$

$$2at_3 = 2nct_1 + b(n-1)t_2$$

$$3at_4 = 3ndt_1 + c(2n-1)t_2 + b(n-2)t_3$$

$$4at_5 = 4net_1 + d(3n-1)t_2 + c(2n-2)t_3 + b(n-3)t_4$$

$$5at_6 = 5nft_1 + e(4n-1)t_2 + d(3n-2)t_3 + c(2n-3)t_4 + b(n-4)t_5,$$

etc. *etc.*

where the law is manifest, and since the first term, t_1, of the expansion is obviously $= a^n$, the preceding set of equations, by substituting in succession the values of t_1, t_2, t_3, ... will give

$$t_2 = nba^{n-1}$$

$$2t_3 = 2nca^{n-1} + n(n-1)b^2a^{n-2}$$

$$2.3t_4 = 2.3nda^{n-1} + 3n(n-1)2bca^{n-2} + n(n-1)(n-2)b^3a^{n-3}$$

$$2.3.4t_5 = 2.3.4nea^{n-1} + 3.4n(n-1)(2bd + c^2)a^{n-2}$$
$$+ 4n(n-1)(n-2)3b^2ca^{n-3} + n(n-1)(n-2)(n-3)b^4a^{n-4},$$

etc. *etc.*

Substituting in (1) the values of t_1, t_2, t_3, *etc.*, derived from these equations, we have finally

$$(a + bx + cx^2 + dx^3 + ex^4 + \ldots\ldots)^n = a^n$$
$$+ nba^{n-1}x$$
$$+ \left\{nca^{n-1} + \frac{n(n-1)}{1.2}b^2a^{n-2}\right\}x^2$$
$$+ \left\{nda^{n-1} + \frac{n(n-1)}{1.2}2bca^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}b^3a^{n-3}\right\}x^3$$
$$+ \left\{\begin{array}{r} nea^{n-1} + \frac{n(n-1)}{1.2}(2bd+c^2)a^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}3b^2ca^{n-3} \\ + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}b^4a^{n-4} \end{array}\right\}x^4$$

+ *etc.* *etc.*

The law of the coefficients of the several powers of x in the preceding expansion is very obvious, with the exception of the partial coefficients of the several powers of a, which involve the combinations of b, c, d, *etc.* The subjoined scheme will shew how these factors which occur in the coefficients of the different powers of a may be obtained at pleasure.

It is worthy of remark, that the sum of the indices of the powers of a, b, c, d, etc., in each of the partial coefficients of the several powers of x, is always equal to n; and thus we know in what vertical column of the table to look for any required coefficient, as in example 3, p. 184.

Table of factors in the coefficients of the several powers of a.

	a^{n-1}	a^{n-2}	a^{n-3}	a^{n-4}	a^{n-5}
b	$b.b$				
c	$c.b$ $b.c$	$b.b^2$			
d	$d.b$ $c.c$ $b.d$	$c.b^2$ $b.2bc$	$b.b^3$		
e	$e.b$ $d.c$ $c.d$ $b.e$	$d.b^2$ $c.2bc$ $b(2bd+c^2)$	$c.b^3$ $b.3b^2c$	$b.b^4$	
f	$f.b$ $e.c$ $d.d$ $c.e$ $b.f$	$e.b^2$ $d.2bc$ $c(2bd+c^2)$ $b(2be+2cd)$	$d.b^3$ $c.3b^2c$ $b(3b^2d+3bc^2)$	$c.b^4$ $b.4b^3c$	$b.b^5$

The preceding table may readily be continued to any extent, by observing that the *second factor of the last line in any division of the table is the sum of all the products in the next higher division of the preceding vertical column.* For example, the second factors in the last line of the horizontal column (f), are obtained from the sums of the products in the horizontal column (e) in this manner:

$eb+dc+cd+be=2be+2cd$; $d.b^2+c.2bc+b(2bd+c^2)=3b^2d+3bc^2$;

$c.b^3+b.3b^2c=4b^3c$, $b.b^4=b^5$. Thus the table may be continued at pleasure, and the expansion of the multinomial carried to any extent, recollecting that only the *second factors in the last line of any horizontal column* are to be used in the coefficients of the respective powers of a, to which they belong, so that the second factors in the last line of (f), in connection with the corresponding power of a, are to be used in finding the whole coefficient of x^5, and so on.

EXAMPLES.

1. Find the terms of the expansion of $(1 + x + x^2 + x^3 + \ldots.)^4$.

Here $a = b = c = d = \ldots = 1$, and $n = 4$; hence by the formula

$$(1+x+x^2+x^3+x^4+\ldots.)^4 = 1+4x+10x^2+20x^3+35x^4+56x^5+\ldots$$

2. Find the coefficient of x^5 in the expansion of $(1 + 2x + 3x^2 + 4x^3 + \ldots.)^6$.

Here $a = 1$, $b = 2$, $c = 3$, $d = 4$, *etc.*, and $n = 6$, then the second factors of the last line of the horizontal column (f) in the table are

$2be + 2cd = 44$, $3b^2d + 3bc^2 = 102$, $4b^3c = 96$; and $b^5 = 32$;

hence by the formula, the coefficient of x^5 is

$$6.6 + 15.44 + 20.102 + 15.96 + 6.32 = 4368.$$

3. Find the term of $(a + b + c)^{10}$ involving $a^5b^3c^2$.*

Since $n = 10$, the coefficient of b^3c^2 will be found in the second factor of the last line in the vertical column of the table under a^{n-5}. Continuing this column two steps farther by the process already mentioned, we find the second factor to be $5b^4d + 10b^3c^2$; hence by the formula, the complete term is

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} \times 10b^3c^2a^5 = \frac{10.9.8.7.6}{1.2.3.4.5} \times 10a^5b^3c^2 = 2520a^5b^3c^2.$$

4. Expand $(x - x^3 + x^5 - x^7 + \ldots)^2$.

Ans. $x^2 - 2x^4 + 3x^6 - 4x^8 + 5x^{10} - \ldots.$

5. Expand $(1 + x + x^2 + x^3 + x^4 + \ldots.)^{\frac{1}{2}}$.

Ans. $1 + \frac{1}{2}x + \frac{1 \,.\, 3}{2 \,.\, 4}x^2 + \frac{3 \,.\, 5}{2 \,.\, 4 \,.\, 6}x^3 + \ldots..$

6. Find the coefficient of x^5 in the expansion of

$$(a + bx + cx^2 + dx^3)^5.$$

Ans. $20a^3cd + 30a^2b^2d + 30a^2bc^2 + 20ab^3c + b^5$.

7. Required the term involving $a^2b^2c^3$ in the expansion of $(a + b + c)^7$.

Ans. $210a^2b^2c^3$.

---

* In examples of this kind, the general formula is applicable by taking x equal to unity.

TABLE OF FACTORS involving b, c, d, e, etc., in the coefficients of the several powers of a in the expansion of $(a+bx+cx^2+dx^3+ex^4+\ldots)$.

	a^{n-1}	a^{n-2}	a^{n-3}	a^{n-4}	a^{n-5}	a^{n-6}	a^{n-7}
b	$b \cdot b$						
c	$c \cdot b$						
	$b \cdot c$	$b \cdot b^2$					
d	$d \cdot b$						
	$c \cdot c$	$c \cdot b^2$					
	$b \cdot d$	$b \cdot 2bc$	$b \cdot b^3$				
e	$e \cdot b$						
	$d \cdot c$	$d \cdot b^2$					
	$c \cdot d$	$c \cdot 2bc$	$c \cdot b^3$				
	$b \cdot e$	$b(2bd+c^2)$	$b \cdot 3b^2c$	$b \cdot b^4$			
f	$f \cdot b$						
	$e \cdot c$	$e \cdot b^2$					
	$d \cdot d$	$d \cdot 2bc$	$d \cdot b^3$				
	$c \cdot e$	$c(2bd+c^2)$	$c \cdot 3b^2c$	$c \cdot b^4$			
	$b \cdot f$	$b(2be+2cd)$	$b(3b^2d+3bc^2)$	$b \cdot 4b^3c$	$b \cdot b^5$		

TABLE of FACTORS, *etc.*—*Continued.*

	a^{n-1}	a^{n-2}	a^{n-3}	a^{n-4}	a^{n-5}	a^{n-6}	a^{n-7}
g	$g \,.\, b$						
	$f \,.\, c$	$f \,.\, b^2$					
	$e \,.\, d$	$e \,.\, 2bc$	$e \,.\, b^3$				
	$d \,.\, e$	$d(2bd+c^2)$	$d \,.\, 3b^2c$	$d \,.\, b^4$			
	$c \,.\, f$	$c(2be+2cd)$	$c(3b^2d+3bc^2)$	$c \,.\, 4b^3c$	$c \,.\, b^5$		
	$b \,.\, g$	$b(2bf+2ce+d^2)$	$b(3b^2e+6bcd+c^3)$	$b(4b^3d+6b^2c^2)$	$b \,.\, 5b^4c$	$b \,.\, b^6$	
h	$h \,.\, b$						
	$g \,.\, c$	$g \,.\, b^2$					
	$f \,.\, d$	$f \,.\, 2bc$	$f \,.\, b^3$				
	$e \,.\, e$	$e(2bd+c^2)$	$e \,.\, 3b^2c$	$e \,.\, b^4$			
	$d \,.\, f$	$d(2be+2cd)$	$d(3b^2d+6bc^2)$	$d \,.\, 4b^3c$	$d \,.\, b^5$		
	$c \,.\, g$	$c(2bf+2ce+d^2)$	$c(3b^2e+6bcd+c^3)$	$c(4b^3d+6b^2c^2)$	$c \,.\, 5b^4c$	$c \,.\, b^6$	
	$b \,.\, h$	$b(2bg+2cf+2de)$	$b(3b^2f+6bce+3bd^2+3c^2d)$	$b(4b^3e+12b^2cd+4bc^3)$	$b(5b^4d+10b^3c^2)$	$b \,.\, 6b^5c$	$b \,.\, b^7$

EXPONENTIAL THEOREM.

To expand a^x in a series ascending according to the powers of x.

Let $a = 1 + A$, then $A = a - 1$, and by the Binomial Theorem,

$$a^x = (1+A)^x = 1 + xA + \frac{x(x-1)}{1\,.\,2}A^2 + \frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}A^3 + \ldots\ldots$$

$$= 1 + (A - \tfrac{1}{2}A^2 + \tfrac{1}{3}A^3 - \tfrac{1}{4}A^4 + \ldots\ldots)x + A_2x^2 + A_3x^3 + \ldots\ldots$$

where A_2, A_3, ... denote the values of the coefficients of x^2, x^3,...... in terms of A.

Now if $A_1 = A - \tfrac{1}{2}A^2 + \tfrac{1}{3}A^3 - \tfrac{1}{4}A^4 + \ldots\ldots$

$$= (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \ldots\ldots,$$

then $\quad a^x = 1 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \ldots\ldots\ldots\ldots$

Similarly, $a^y = 1 + A_1y + A_2y^2 + A_3y^3 + A_4y^4 + \ldots\ldots\ldots\ldots,$

and $\quad a^{x+y} = 1 + A_1(x+y) + A_2(x+y)^2 + A_3(x+y)^3 + A_4(x+y)^4 + \ldots.$

$$\left.\begin{array}{rrrrrl}
= 1 + A_1x + & A_2x^2 + & A_3x^3 + & A_4x^4 & + \ldots \\
+ A_1y + & 2A_2xy + & 3A_3x^2y + & 4A_4x^3y & + \ldots \\
 & + A_2y^2 + & 3A_3xy^2 + & 6A_4x^2y^2 & + \ldots \\
 & & + A_3y^3 + & 4A_4xy^3 & + \ldots \\
 & & & + A_4y^4 & + \ldots
\end{array}\right\} \ldots\ldots(1)$$

But

$$= a^x \times a^y$$

$$= (1 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + ..)(1 + A_1y + A_2y^2 + A_3y^3 + A_4y^4 + ..)$$

$$\left.\begin{array}{rrrrrl}
= 1 + A_1x + & A_2x^2 + & A_3x^3 + & A_4x^4 & + \ldots\ldots \\
+ A_1y + & A^2_1xy + & A_1A_2x^2y + & A_1A_3x^3y & + \ldots\ldots \\
 & + A_2y^2 + & A_1A_2xy^2 + & A_2{}^2x^2y^2 & + \ldots\ldots \\
 & & + A_3y^3 + & A_1A_3xy^3 & + \ldots\ldots \\
 & & & + A_4y^4 & + \ldots\ldots
\end{array}\right\} \ldots\ldots\ldots(2)$$

Now these two expansions of a^{x+y} must be identical; hence equating the coefficients of like combinations of x and y in (1) and (2), we have

$$2A_2 = A^2_1, \quad 3A_3 = A_1A_2, \quad 4A_4 = A_1A_3, \quad \textit{etc.};$$

$$\therefore A_2 = \frac{A^2_1}{1\,.\,2}, \quad A_3 = \frac{A^3_1}{1\,.\,2\,.\,3}, \quad A_4 = \frac{A^4_1}{1\,.\,2\,.\,3\,.\,4}, \textit{etc.};$$

hence $a^x = 1 + A_1x + \frac{A^2_1x^2}{1.2} + \frac{A^3_1x^3}{1.2.3} + \frac{A^4_1x^4}{1.2.3.4} + \ldots\ldots\ldots;$

and dropping the suffix of A as no longer required, we have

$$a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \frac{A^4x^4}{1.2.3.4} + \frac{A^5x^5}{1.2.3.4.5} + \ldots\ldots;$$

where $A = a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \frac{1}{5}(a-1)^5 - \ldots\ldots$

If e be that value of a which renders $A = 1$; then

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \ldots\ldots\ldots\ldots,$$

and if $x = 1$, we get

$$e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots\ldots = 2{\cdot}718281828459.$$

THEORY OF LOGARITHMS.

A *logarithm* of a number is the index of the power to which a given quantity must be raised that the power may be equal to the number.

Thus, if the power a^x be equal to the number N, or $a^x = N$; then the index x, of the power a^x, is called the logarithm of the number N to the base a, and this is usually written $x = \log_a N$.

A system of logarithms is a series of values of x corresponding to *different* values of N, the base a remaining the *same*. Thus since

$$2^2 = 4, \qquad 2^3 = 8, \qquad 2^5 = 32, \qquad 2^{10} = 1024;$$

$$\therefore\ 2 = \log_2 4,\ 3 = \log_2 8,\ 5 = \log_2 32,\ 10 = \log_2 1024;$$

and by giving N different values, there would be corresponding values to the index x. These values of x, being tabulated, would constitute a system of logarithms to the base 2. Again, since

$$2^6 = 64, \qquad 4^3 = 64, \qquad 8^2 = 64, \qquad 64^1 = 64;$$

$$\therefore\ \log_2 64 = 6,\ \log_4 64 = 3,\ \log_8 64 = 2,\ \log_{64} 64 = 1;$$

consequently there may be different systems of logarithms by merely giving different values to the base, for the logarithm of 64 to the bases 2, 4, 8, 64 are respectively 6, 3, 2, 1.

Since $a^1 = a$, therefore $\log_a a = 1$, and since $e^1 = e$; therefore $\log_e e = 1$; hence in every system of logarithms, the logarithm of its base is 1.

Since $a^0 = 1$ and $e^0 = 1$, therefore in every system of logarithms, the logarithm of 1 is 0.

Since $a^{-\infty} = \dfrac{1}{a^{\infty}} = 0$; therefore the logarithm of 0 is $-\infty$.

If $a^x = -\text{N}$, and a is a real quantity, there is no value of x, positive or negative, which would fulfil the equation; hence the logarithm of a negative number is imaginary.

Properties of Logarithms.

(1) *The logarithm of the product of any number of factors is equal to the sum of the logarithms of those factors.*

Let P, Q, R, S be any quantities, and x, y, z, v their logarithms to any base a; then

$$a^x = \text{P}, \qquad a^y = \text{Q}, \qquad a^z = \text{R}, \qquad a^v = \text{S};$$

$$\therefore\ x = \log_a \text{P}, \quad y = \log_a \text{Q}, \quad z = \log_a \text{R}, \quad v = \log_a \text{S};$$

$$\therefore\ \text{PQRS} = a^x . a^y . a^z . a^v = a^{x+y+z+v};\ \text{hence}$$

$$\log_a \text{PQRS} = x+y+z+v = \log_a \text{P} + \log \text{Q} + \log_a \text{R} + \log_a \text{S}.$$

(2) *The logarithm of a fractional quantity is equal to the logarithm of the numerator minus the logarithm of the denominator.*

Let $a^x = P$ and $a^y = Q$, then $x = \log_a P$ and $y = \log_a Q$; hence

$$\frac{P}{Q} = \frac{a^x}{a^y} = a^{x-y};$$

$$\therefore \log_a \frac{P}{Q} = x - y = \log_a P - \log_a Q.$$

(3) *The logarithm of any power of a quantity is equal to the logarithm of the quantity multiplied by the index of the power.*

Let $a^x = P$; then $x = \log_a P$, and $a^{nx} = P^n$; therefore

$$\log_a P^n = nx = n \log_a P.$$

(4) *The logarithm of any root of a quantity is equal to the logarithm of the quantity divided by the index of the root.*

Let $a^x = P$, then $x = \log_a P$ and $a^{\frac{x}{n}} = P^{\frac{1}{n}}$; therefore

$$\log_a P^{\frac{1}{n}} = \frac{x}{n} = \frac{1}{n} . \log_a P.$$

Hence also $\log P^{\frac{m}{n}} = \frac{m}{n} \log P$.

(5) *The logarithms of any quantity in two different systems are inversely as the logarithms of their bases in any system.*

Let N denote any number or quantity, and a, b any two bases; hence if $a^x = N$ and $b^y = N$; then $a^x = b^y$, and the logarithms of both members to any base, give by (3)

$$x \log a = y \log b, \text{ or } x : y :: \log b : \log a;$$

but $x = \log_a N$, and $y = \log_b N$; consequently

$$\log_a N : \log_b N :: \log b : \log a.$$

These general properties of logarithms are of the highest importance, and the latter (5) will enable us to convert one system of logarithms into another system with great facility. Thus, since

$$\log_a N : \log_b N :: \log_a b : \log_a a,$$

$$\text{and } \log_a a = 1 \text{; therefore } \log_b N = \frac{1}{\log_a b} \cdot \log_a N.$$

Hence, if a system of logarithms be calculated to the base a, a system of logarithms to the base b, may readily be found by means of the preceding equation which expresses the relation between the two systems. The factor $\frac{1}{\log_a b}$ by which the system whose base is a is transformed to the system whose base is b, is called the *modulus* of the system whose base is b.

Computation of Logarithms.

Referring to the exponential theorem, p. 188, we have seen that

$$a^x = 1 + Ax + \frac{A^2x^2}{1.\ 2} + \frac{A^3x^3}{1\ .\ 2\ .\ 3} + \cdots,$$

$$\text{where } A = a-1-\tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \ldots.$$

Now, let $x = \frac{1}{A}$; then $a^{\frac{1}{A}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots.$;

but the second member has been shewn (p. 188) to be equal to e, hence

$$a^{\frac{1}{A}} = e, \text{ and therefore } a = e^A \text{; hence}$$

$$\log_e a = A = a - 1 - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \ldots..$$

Let now $a - 1 = z$; then $a = 1 + z$, and we have

$$\log_e (1 + z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 + \tfrac{1}{5}z^5 - \ldots\ldots (1)$$

Writing $-z$ for z gives

$$\log_e (1-z) = -z - \tfrac{1}{2}z^2 - \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 - \tfrac{1}{5}z^5 - \ldots (2)$$

Subtracting (2) from (1), and recollecting that

$$\log (1 + z) - \log (1-z) = \log \frac{1+z}{1-z},$$

$$\therefore \log_e \frac{1+z}{1-z} = 2(z + \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5 + \tfrac{1}{7}z^7 + \ldots\ldots.) \ldots.. (3)$$

If $\frac{1+z}{1-z}=\frac{n+x}{n}$; then $z=\frac{x}{2n+x}$, and the last equation becomes

$$\log_e\frac{n+x}{n}=2\left\{\frac{x}{2n+x}+\tfrac{1}{3}\left(\frac{x}{2n+x}\right)^3+\tfrac{1}{5}\left(\frac{x}{2n+x}\right)^5+\ldots\ldots\right\}\ldots(4);$$

hence by the properties of logarithms (5) we have

$$\log_a\frac{n+x}{n}=2\mathrm{M}_a\left\{\frac{x}{2n+x}+\tfrac{1}{3}\left(\frac{x}{2n+x}\right)^3+\tfrac{1}{5}\left(\frac{x}{2n+x}\right)^5+\ldots\right\}\ldots(5);$$

where $\mathrm{M}_a=\frac{1}{\log_e a}=$ modulus of the system whose base is a.

From the equations (4) and (5) we have, since $\log\frac{n+x}{n}=\log(n+x)-\log n$,

$$\log_e(n+x)=\log_e n+2\left\{\frac{x}{2n+x}+\tfrac{1}{3}\left(\frac{x}{2n+x}\right)^3+\tfrac{1}{5}\left(\frac{x}{2n+x}\right)^5+.\right\}\ldots(6)$$

$$\log_a(n+x)=\log_a n+2\mathrm{M}_a\left\{\frac{x}{2n+x}+\tfrac{1}{3}\left(\frac{x}{2n+x}\right)^3+\tfrac{1}{5}\left(\frac{x}{2n+x}\right)^5+.\right\}\ldots(7)$$

Let $\frac{1+z}{1-z}=\frac{\mathrm{B}}{b}$, and consequently $z=\frac{\mathrm{B}-b}{\mathrm{B}+b}$, then substituting this value of z in formula (3), there results

$$\log_a\frac{\mathrm{B}}{b}=2\mathrm{M}_a\left\{\frac{\mathrm{B}-b}{\mathrm{B}+b}+\tfrac{1}{3}\left(\frac{\mathrm{B}-b}{\mathrm{B}+b}\right)^3+\tfrac{1}{5}\left(\frac{\mathrm{B}-b}{\mathrm{B}+b}\right)^5+etc.\ldots\right\}\ldots(8)$$

Let $z=\frac{2}{p^3-3p}$, substituting this in formula (3), then

$$\frac{1+z}{1-z}=\frac{1+\frac{2}{p^3-3p}}{1-\frac{2}{p^3-3p}}=\frac{p^3-3p+2}{p^3-3p-2}=\frac{(p-1)^2(p+2)}{(p+1)^2(p-2)};\text{ whence}$$

$$2\log_a(p-1)+\log_a(p+2)-2\log_a(p+1)-\log_a(p-2)=$$

$$2\mathrm{M}_a\left\{\frac{2}{p^3-3p}+\tfrac{1}{3}\left(\frac{2}{p^3-3p}\right)^3+\tfrac{1}{5}\left(\frac{2}{p^3-3p}\right)^5+etc.\ldots\right\}\ldots\ldots\ldots(9)$$

which by transposition becomes

$$\mathrm{Log}_a(p+2) = \log_a(p-2) + 2\log_a(p+1) - 2\log_a(p-1) +$$

$$2\mathrm{M}_a\left\{\frac{2}{p^3-3p} + \tfrac{1}{3}\left(\frac{2}{p^3-3p}\right)^3 + \tfrac{1}{5}\left(\frac{2}{p^3-3p}\right)^5 + etc.\right\}\ldots\ldots(10).$$

This is the formula of Borda, which, when p exceeds 3, is very convergent.

If, in formula (7) $n = x^4 - 25x^2 \quad = x^2(x+5)(x-5)$,
and $n + x = x^4 - 25x^2 + 144 = (x+3)(x-3)(x+4)(x-4)$;

$$\mathrm{Log}_a(x+5) = \{\log_a(x+3) + \log_a(x-3) + \log_a(x+4) + \log_a(x-4)\}$$

$$-\left\{\log_a(x-5) + 2\log_a x\right\}$$

$$-2\mathrm{M}_a\left\{\frac{72}{x^4-25x^2+72} + \tfrac{1}{3}\left(\frac{72}{x^4-25x^2+72}\right)^3 + etc.\ldots(11).\right.$$

This is the formula of Haros, which is very convergent when the value of x becomes somewhat great.

These two last formulæ (10) and (11) are very useful in examining the accuracy of logarithmic tables already computed, or in checking the accuracy of those new ones in the course of computation. To these might have been added that of Lavernède; but our limits will not permit.

Let N be the number of which i is the index, and d the mantissa or decimal part of the logarithm; then $\log_a \mathrm{N} = i + d$; and $\log_a 10^m\mathrm{N} = (i + m) + d$; whence it is evident that the decimal part of a logarithm remains the same when the significant figures are constant.

The series in (6) or (7) is that usually employed in computing logarithms; but before we can compute a system of logarithms to any other base than e, whose value has already been found to be

2·718281828459...(p. 188), it is obvious that we must previously determine the value of the modulus M_a or $\frac{1}{\log_e a}$ by means of the series in (6). We shall therefore proceed to compute the logarithms of one or two numbers to the base e. The system of logarithms to the base e is called the *Napierian system*, from the name of the inventor of logarithms, Lord Napier of Merchiston, in Scotland, who first published this system in 1614.

1. Calculate the logarithm of 2 to the Napierian base.

In the formula (6), let $n = 1$, and $x = 1$; then, since $\log 1 = 0$,

$$\log_e 2 = 2\{\tfrac{1}{3} + \tfrac{1}{3}(\tfrac{1}{3})^3 + \tfrac{1}{5}(\tfrac{1}{3})^5 + \tfrac{1}{7}(\tfrac{1}{3})^7 + \ldots\ldots\}.$$

$$\begin{array}{llcr}
3\,)\ 1\cdot0000000000 & & & \\
9\,)\ \cdot3333333333 = \frac{1}{3}; & \therefore & \frac{1}{3} = & \cdot3333333333 \\
9\,)\ \cdot0370370370 = (\frac{1}{3})^3; & \therefore & \frac{1}{3}(\frac{1}{3})^3 = & \cdot0123456790 \\
9\,)\ \cdot0041152263 = (\frac{1}{3})^5; & \therefore & \frac{1}{5}(\frac{1}{3})^5 = & \cdot0008230453 \\
9\,)\ \cdot0004572474 = (\frac{1}{3})^7; & \therefore & \frac{1}{7}(\frac{1}{3})^7 = & \cdot0000653211 \\
9\,)\ \cdot0000508053 = (\frac{1}{3})^9; & \therefore & \frac{1}{9}(\frac{1}{3})^9 = & \cdot0000056450 \\
9\,)\ \cdot0000056450 = (\frac{1}{3})^{11}; & \therefore & \frac{1}{11}(\frac{1}{3})^{11} = & \cdot0000005132 \\
9\,)\ \cdot0000006272 = (\frac{1}{3})^{13}; & \therefore & \frac{1}{13}(\frac{1}{3})^{13} = & \cdot0000000483 \\
9\,)\ \cdot0000000697 = (\frac{1}{3})^{15}; & \therefore & \frac{1}{15}(\frac{1}{3})^{15} = & \cdot0000000046 \\
9\,)\ \cdot0000000077 = (\frac{1}{3})^{17}; & \therefore & \frac{1}{17}(\frac{1}{3})^{17} = & \cdot0000000005 \\
\hline
 & & & \cdot3465735903 \\
 & & & 2 \\
\hline
 & \therefore & \log_e 2 = & \cdot6931471806 \\
 & & & 2 \\
\hline
\end{array}$$

But $\log 4 = \log 2^2 = 2 \log 2$; $\therefore\ \log_e 4 = 1\cdot3862943612$

2. To calculate the Napierian logarithm of 5.

Let $x = 1$ and $n = 4$; then by (6)

$$\log_e 5 = \log_e 4 + 2\{\tfrac{1}{9} + \tfrac{1}{3}(\tfrac{1}{9})^3 + \tfrac{1}{5}(\tfrac{1}{9})^5 + \tfrac{1}{7}(\tfrac{1}{9})^7 + \ldots\}$$

$$\begin{array}{lllr}
9\,)\,1\cdot0000000000 & & & \\
\hline
9\,)\,\cdot1111111111 = \frac{1}{9} & \therefore & \frac{1}{9} = & \cdot1111111111 \\
\hline
9\,)\,\cdot0123456790 & & & \\
\hline
9\,)\,\cdot0013717421 = (\frac{1}{9})^3 & \therefore & \frac{1}{3}(\frac{1}{9})^3 = & \cdot0004572474 \\
\hline
9\,)\,.0001524158 & & & \\
\hline
9\,)\,\cdot0000169351 = (\frac{1}{9})^5 & \therefore & \frac{1}{5}(\frac{1}{9})^5 = & \cdot0000033870 \\
\hline
9\,)\,\cdot0000018817 & & & \\
\hline
9\,)\,\cdot0000002091 = (\frac{1}{9})^7 & \therefore & \frac{1}{7}(\frac{1}{9})^7 = & \cdot0000000299 \\
\hline
9\,)\,\cdot0000000232 & & & \\
\hline
\cdot0000000026 = (\frac{1}{9})^9 & \therefore & \frac{1}{9}(\frac{1}{9})^9 = & \cdot0000000003 \\
\cline{4-4}
 & & & \cdot1115717757 \\
 & & & 2 \\
\cline{4-4}
 & & & \cdot2231435514 \\
 & & \log_e 4 = & 1\cdot3862943608 \\
\cline{4-4}
 & & \therefore \log_e 5 = & 1\cdot6094379122.
\end{array}$$

Now by the properties of logarithms (1) we have log 10 = log (2 × 5) = log 2 + log 5; therefore

$$\begin{array}{rr}
\log_e 2 = & \cdot6931471806 \\
\log_e 5 = & 1\cdot6094379122 \\
\hline
\text{and } \log_e 10 = & 2\cdot3025850928.
\end{array}$$

We can now compute logarithms to the bases 2, 4, 5, and 10, since the values of the moduli for all these bases can readily be determined from the preceding computations. Thus for the base 10, we have

$$M_{10} = \frac{1}{\log_e 10} = \frac{1}{2\cdot3025850928} = \cdot43429448190$$

The system of logarithms whose base is 10, was calculated by Henry Briggs, Professor of Geometry in the University of Oxford, and is that which is now in common use. This system is hence called the *Briggean* or *common* system, on account of the coincidence

TABLE of FACTORS, *etc.*—*Continued.*

	a^{n-1}	a^{n-2}	a^{n-3}	a^{n-4}	a^{n-5}	a^{n-6}	a^{n-7}
g	$g.b$						
	$f.c$	$f.b^2$					
	$e.d$	$e.2bc$	$e.b^3$				
	$d.e$	$d(2bd+c^2)$	$d.3b^2c$	$d.b^4$			
	$c.f$	$c(2be+2cd)$	$c(3b^2d+3bc^2)$	$c.4b^3c$	$c.b^5$		
	$b.g$	$b(2bf+2ce+d^2)$	$b(3b^2e+6bcd+c^3)$	$b(4b^3d+6b^2c^2)$	$b.5b^4c$	$b.b^6$	
h	$h.b$						
	$g.c$	$g.b^2$					
	$f.d$	$f.2bc$	$f.b^3$				
	$e.e$	$e(2bd+c^2)$	$e.3b^2c$	$e.b^4$			
	$d.f$	$d(2be+2cd)$	$d(3b^2d+6bc^2)$	$d.4b^3c$	$d.b^5$		
	$c.g$	$c(2bf+2ce+d^2)$	$c(3b^2e+6bcd+c^3)$	$c(4b^3d+6b^2c^2)$	$c.5b^4c$	$c.b^6$	

EXPONENTIAL THEOREM.

To expand a^x in a series ascending according to the powers of x.

Let $a = 1 + \text{A}$, then $\text{A} = a - 1$, and by the Binomial Theorem,

$$a^x = (1+\text{A})^x = 1 + x\text{A} + \frac{x(x-1)}{1\,.\,2}\text{A}^2 + \frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}\text{A}^3 + \ldots\ldots$$

$$= 1 + (\text{A} - \tfrac{1}{2}\text{A}^2 + \tfrac{1}{3}\text{A}^3 - \tfrac{1}{4}\text{A}^4 + \ldots\ldots)x + \text{A}_2x^2 + \text{A}_3x^3 + \ldots\ldots$$

where A_2, A_3, ... denote the values of the coefficients of x^2, x^3,...... in terms of A.

Now if $\text{A}_1 = \text{A} - \tfrac{1}{2}\text{A}^2 + \tfrac{1}{3}\text{A}^3 - \tfrac{1}{4}\text{A}^4 + \ldots\ldots$

$$= (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \ldots\ldots,$$

then $\quad a^x = 1 + \text{A}_1x + \text{A}_2x^2 + \text{A}_3x^3 + \text{A}_4x^4 + \ldots\ldots\ldots\ldots$

Similarly, $a^y = 1 + \text{A}_1y + \text{A}_2y^2 + \text{A}_3y^3 + \text{A}_4y^4 + \ldots\ldots\ldots\ldots,$

and $\quad a^{x+y} = 1 + \text{A}_1(x+y) + \text{A}_2(x+y)^2 + \text{A}_3(x+y)^3 + \text{A}_4(x+y)^4 + \ldots.$

$$\left.\begin{array}{rl} = 1 + \text{A}_1x + \text{A}_2x^2 + \text{A}_3x^3 + \text{A}_4x^4 & + \ldots \\ + \text{A}_1y + 2\text{A}_2xy + 3\text{A}_3x^2y + 4\text{A}_4x^3y & + \ldots \\ + \text{A}_2y^2 + 3\text{A}_3xy^2 + 6\text{A}_4x^2y^2 & + \ldots \\ + \text{A}_3y^3 + 4\text{A}_4xy^3 & + \ldots \\ + \text{A}_4y^4 & + \ldots \end{array}\right\} \ldots\ldots(1)$$

But

$$= a^x \times a^y$$

$$= (1 + \text{A}_1x + \text{A}_2x^2 + \text{A}_3x^3 + \text{A}_4x^4 + ..)(1 + \text{A}_1y + \text{A}_2y^2 + \text{A}_3y^3 + \text{A}_4y^4 + ..)$$

$$\left.\begin{array}{rl} = 1 + \text{A}_1x + \text{A}_2x^2 + \text{A}_3x^3 + \text{A}_4x^4 & + \ldots\ldots \\ + \text{A}_1y + \text{A}^2{}_1xy + \text{A}_1\text{A}_2x^2y + \text{A}_1\text{A}_3x^3y & + \ldots\ldots \\ + \text{A}_2y^2 + \text{A}_1\text{A}_2xy^2 + \text{A}_2{}^2x^2y^2 & + \ldots\ldots \\ + \text{A}_3y^3 + \text{A}_1\text{A}_3xy^3 & + \ldots\ldots \\ + \text{A}_4y^4 & + \ldots\ldots \end{array}\right\} \ldots\ldots\ldots (2)$$

Rule. If the number have n digits before the decimal point, the characteristic is $n - 1$; but if the number be a decimal, and have n ciphers before the first significant digit, the characteristic is $-(n + 1)$

TABLE OF LOGARITHMS OF PRIME NUMBERS BETWEEN 1 AND 100.

Nos.	Logarithms.	Nos.	Logarithms.	Nos.	Logarithms.
2	0·3010300	29	1·4623980	61	1·7853298
3	0·4771213	31	1·4913617	67	1·8260748
7	0·8450980	37	1·5682017	71	1·8512583
11	1·0413927	41	1·6127839	73	1·8633229
13	1·1139434	43	1·6334685	79	1·8976271
17	1·2304489	47	1·6720979	83	1·9190781
19	1·2787536	53	1·7242759	89	1·9493900
23	1·3617278	59	1·7708520	97	1·9867717

From the preceding brief exposition of the theory of logarithms the student will be enabled to compute the logarithms here given, and thence deduce the logarithms of all numbers from 1 to 100, as well as to pursue the subject to any extent.

EXAMPLES FOR PRACTICE.

1. Given the common logarithms of 2 and 3, to find the logarithms of 5, 12, 18, 22·5 and ·0072.

Since $5 = \frac{10}{2}$, $\therefore \log 5 = \log 10 - \log 2 = 1 - \log 2 = \cdot 6989700$

$12 = 3 \times 2^2, \therefore \log 12 = \log 3 + 2 \log 2 \quad = 1\cdot 0791812$

$18 = 2 \times 3^2, \therefore \log 18 = \log 2 + 2 \log 3 \quad = 1\cdot 2552725$

$$22\cdot 5 = \frac{90}{4} = \frac{10 \times 3^2}{2^2}, \therefore \log 22\cdot 5 = 1 + 2 \log 3 - 2 \log 2 = 1\cdot 3521825$$

$$\cdot 0072 = \frac{72}{10^4} = \frac{2^3 \times 3^2}{10^4}, \therefore \log \cdot 0072 = 3 \log 2 + 2 \log 3 - 4 = \bar{3}\cdot 8573325.$$

2. Given the logarithms of 3 and 7, to find the logarithms of ·21 and 14700.

Ans. $\bar{1}\cdot 3222193$, and $4\cdot 1673173$.

3. Given the logarithms of 2 and 3, to find the logarithms of $\frac{9}{16}$ and $\frac{4}{375}$.

Ans. $2 \log 3 - 4 \log 2 = \bar{1}\cdot7501225$
and $5 \log 2 - \log 3 - 3 = \bar{2}\cdot0280287$.

4. Find the logarithm of 180 to the base 6, or solve the equation $6^x = 180$.

$$\text{Ans. } x = \frac{1 + \log 2 + 2 \log 3}{\log 2 + \log 3} = 2\cdot8982444.$$

5. Find the logarithm of $\frac{2}{3}$ in the system whose base is $\frac{5}{7}$.

Ans. 1·2050476.

6. Calculate the common logarithms of 17 and 19.

Ans. 1·2304489 and 1·2787536.

THEORY OF EQUATIONS.

The form of an equation of the nth degree in x, is

$$x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + \ldots\ldots + t = 0 \ldots\ldots (\alpha),$$

but to facilitate the following investigations, we shall denote the first member of equation (α), which is a function of x, by fx, and thus the general form may be written $fx = 0$.

Prop. I. *If a be a root of the equation (α), or $fx = 0$, then fx is divisible by $x - a$ without remainder.*

To simplify the demonstration, let the equation be one of the fourth degree, as

$$x^4 + px^3 + qx^2 + rx + s = 0 \ldots .(1),$$

and let it be divided by $x - a$, by the synthetic mode of division, (p. 24),

$$\begin{array}{r|l} & 1 + p + q \;\; + r \;\; + s \\ +a & \quad\;\; a + ap_1 + aq_1 + ar_1 \\ \hline & 1 + p_1 + q_1 + r_1 + s_1, \end{array}$$

where $p_1 = a + p$, $q_1 = ap_1 + q$, $r_1 = aq_1 + r$, and the remainder $s_1 = ar_1 + s$; hence

$$\begin{aligned} s_1 &= a(aq_1 + r) + s & &= a^2q_1 + ar + s \\ &= a^2(ap_1 + q) + ar + s & &= a^3p_1 + a^2q + ar + s \\ &= a^3(a + p) + a^2q + ar + s & &= a^4 + pa^3 + qa^2 + ra + s. \end{aligned}$$

Now, a being a root of equation (1), that equation will be verified by the substitution of a for x; therefore $a^4 + pa^3 + qa^2 + ra + s = 0$, and consequently the remainder s_1 is also $= 0$, which proves the proposition in this case, and the same mode of proof may be extended to the general form in eq. (α).

PROP. II. *The first member of equation (α) is the product of n factors of the first degree.*

Let a_1 be a root of eq. (α), or of $fx = 0$: then, by the last proposition, we have

$$fx = (x - a_1)f'x = 0.$$

If a_2 be a second root of eq. (α), then fx or $(x - a_1)f'x$ is divisible without remainder, by $x - a_2$; but $x - a_1$ is not divisible by $x - a_2$; hence $f'x$, which is an expression involving the $(n - 1)$th power of x, must be divisible by $x - a_2$; therefore

$$fx = (x - a_1)(x - a_2)f''x = 0;$$

where $f''x$ is a polynomial in which x^{n-2} is the highest power of x. Proceeding in this manner, it will be seen that, if $a_3, a_4, \ldots a_n$ are roots of eq. (α),

$$fx = (x - a_1)(x - a_2)(x - a_3) \ldots . (x - a_n) = 0 \ldots . (2).$$

Hence, since there are n factors of the first degree, the equation (α) has as many roots as there are factors, and therefore an equation of the nth degree has n roots. If $a_1 = a_2$, the equation will have two *equal* roots, and so on.

Cor. If one or more roots, $a_1, a_2, \ldots$ of an equation are known, the equation containing the remaining roots will be found by dividing the proposed equation by $x - a_1$, and the resulting equation by $x - a_2$, and so on.

Prop. III. *To form the equation whose roots are* $a_1, a_2, a_3 \ldots\ldots a_n$.

Since the first member of the equation is equal to the continued product of $x - a_1$, $x - a_2$, $x - a_3$, *etc.*, by last proposition; therefore the required equation is

$$(x - a_1)(x - a_2)(x - a_3)\ldots\ldots(x - a_n) = 0.$$

Multiplying the factors $x - a_1$ and $x - a_2$, we shall obtain the quadratic equation whose roots are a_1 and a_2: multiplying this result by the next factor $x - a_3$, we get the cubic equation whose roots are a_1, a_2, a_3; and so on, as in the following scheme: —

$x - a_1$ = first factor.

$x - a_2$ = second factor.

$$\left.\begin{array}{l|l} x^2 - a_1 & x + a_1 a_2 = 0 \\ \quad\;\; - a_2 & \end{array}\right\}\ldots\ldots(1)$$

$x - a_3$ = third factor.

$$\left.\begin{array}{l|l|l} x^3 - a_1 & x^2 + a_1 a_2 & x - a_1 a_2 a_3 = 0 \\ \quad\;\; - a_2 & \quad\;\; + a_1 a_3 & \\ \quad\;\; - a_3 & \quad\;\; + a_2 a_3 & \end{array}\right\}\ldots\ldots\ldots\ldots(2)$$

$x - a_4$ = fourth factor.

$$\left.\begin{array}{l|l|l|l} x^4 - a_1 & x^3 + a_1 a_2 & x^2 - a_1 a_2 a_3 & x + a_1 a_2 a_3 a_4 = 0 \\ \quad\;\; - a_2 & \quad\;\; + a_1 a_3 & \quad\;\; - a_1 a_2 a_4 & \\ \quad\;\; - a_3 & \quad\;\; + a_2 a_3 & \quad\;\; - a_1 a_3 a_4 & \\ \quad\;\; - a_4 & \quad\;\; + a_1 a_4 & \quad\;\; - a_2 a_3 a_4 & \\ & \quad\;\; + a_2 a_4 & & \\ & \quad\;\; + a_3 a_4 & & \end{array}\right\}\ldots\ldots\ldots(3)$$

By continuing this process of multiplication through all the factors, the required equation will be obtained. From this scheme, some important relations between the roots of an equation, and the coefficients of its terms, may be deduced.

(1.) The coefficient of the second term is the sum of the roots with their signs changed.

(2.) The coefficient of the third term is the sum of the products of the roots, taken two and two, with their signs changed.

(3.) The coefficient of the fourth term is the sum of the products of the roots, taken three and three, with their signs changed; and so on, for the other coefficients.

(4.) The last or absolute term is the product of all the roots with their signs changed.

Cor. 1. If the second term of an equation is wanting, the sum of the positive roots is equal to the sum of the negative roots.

Cor. 2. If the signs of the coefficients of the terms of an equation be all positive, all the roots will be negative, and if the signs be alternately positive and negative, all the roots, if they are real, will be positive.

Cor. 3. Every root of an equation is a divisor of the last term.

We may here remark, that surd roots and imaginary roots can enter equations only by pairs; for if the coefficients of an equation do not involve surd or imaginary expressions, the equation cannot have a *single* surd, or *single* imaginary root, because, by this proposition, such surd or imaginary expressions would occur in the coefficients of some of the terms of the equation; but surd or imaginary roots may occur in pairs of the form $a+\sqrt{b}$, $a-\sqrt{b}$; $a+b\sqrt{-1}$, $a-b\sqrt{-1}$; for since

$$\{x-(a+\sqrt{b})\}\ \{x-(a-\sqrt{b})\}=x^2-2ax+a^2-b=(x-a)^2-b$$

$$\{x-(a+b\sqrt{-1})\}\{x-(a-b\sqrt{-1})\}=x^2-2ax+a^2+b^2=(x-a)^2+b^2;$$

therefore no surd or imaginary expressions would be found among the coefficients of the terms of the equation. Want of room prevents the insertion of a formal proof of this important property.

EXAMPLES.

1. Form the equation whose roots are 2, -3, 5, and -6.

Here we must multiply the factors $x-2$, $x+3$, $x-5$ and $x+6$ together; and employing the method of detached coefficients (p. 16), we have this process:

$$\begin{array}{l}
1-2(+3 \\
\quad +3-6 \\
\hline
1+1-6(-5 \\
\quad -5-5+30 \\
\hline
1-4-11+30(+6 \\
\quad +6-24-66+180 \\
\hline
1+2-35-36+180
\end{array}$$

$\therefore x^4+2x^3-35x^2-36x+180=0$ is the equation required.

2. Form the equation whose roots are 1, -1, 2, -3, and 4.
Ans. $x^5-3x^4-11x^3+27x^2+10x-24=0$.

3. Form the equation whose roots are 5, $2+\sqrt{3}$ and $2-\sqrt{3}$.
Ans. $x^3-9x^2+21x-5=0$.

4. Form the equation whose roots are 2, -2, $3+\sqrt{5}$ and $3-\sqrt{5}$.
Ans. $x^4-6x^3+24x-16=0$.

5. Form the equation whose roots are -3, -4, $2+3\sqrt{-1}$ and $2-3\sqrt{-1}$.
Ans. $x^4+3x^3-3x^2+43x+156=0$.

6. Two roots of the equation $4x^4-14x^3-5x^2+31x+6=0$ are 2 and 3; find the equation containing the two other roots.
Ans. $4x^2+6x+1=0$.

Prop. IV. *If the roots of an equation be all real, there will be as many changes or variations of the signs of the successive terms as there are positive roots, and as many permanencies of the signs as there are negative roots.*

A *variation* of sign is a change from + to − or from − to + in two consecutive terms, and a *permanence* is a continuance of sign as

+, +, or −, −, in two consecutive terms. Thus in the series of signs + − + + − −, there are *three* variations and *two* permanencies.

Let $x^4 - px^3 + qx^2 + rx - s = 0$ be an equation, then multiplying it by a factor $x - a$, which corresponds to a positive root, a; and also by a factor $x + b$, which corresponds to a negative root, $-b$; we have the following processes, where the signs of the terms are only employed, viz.:—

$$
\begin{array}{cccccc}
+ & - & + & + & - & \\
+ & - & & & & \\
\hline
+ & - & + & + & - & \\
 & - & + & - & - & + \\
\hline
+ & - & + & \pm & - & +
\end{array}
\qquad\qquad
\begin{array}{cccccc}
+ & - & + & + & - & \\
+ & + & & & & \\
\hline
+ & - & + & + & - & \\
 & + & - & + & + & - \\
\hline
+ & \pm & \pm & + & \pm & -
\end{array}
$$

where the double sign occurs when the quantities to be added have contrary signs, and it is not known which is the greater quantity. Now in the original equation, there are *three* variations and *one* permanence, and in the result of the first process, there is still only *one* permanence, but *four* variations; while in the second process, there are at least *two* permanencies, taking the double sign so as to give the smallest number of permanencies, viz., + − + + + −, or + − + + − −; hence the introduction of a *positive* root increases the number of variations by one, and the introduction of a *negative* root increases the number of permanencies by one. This is true for every root that is introduced, and hence the truth of the proposition is established. The student would see this proof more clearly by taking a *numerical* example or two, as no double sign can then occur.

Conversely, in any equation whose roots are all real, there are as many positive roots as there are variations of sign, and as many negative roots as there are permanencies of sign. Thus the series of signs in the equation $x^4 - 3x^3 - 15x^2 + 49x - 12 = 0$ is + − − + −, indicating *three* positive roots, and *one* negative, since there are *three* variations and *one* permanence.

De Gua's Criterion of Imaginary Roots.

If a term be wanting in an equation, and if the terms on each side of the place of the absent term have like signs, the equation has at least two imaginary roots.

For the sign of the term wanting may be taken either + or −, and the signs + + + indicate two negative roots, while + − + indicate two positive roots. These two roots cannot therefore, by the preceding proposition, be real, and hence they must be imaginary. The same conclusion follows from the other series of signs − ± −. Thus the cubic equation $x^3 + ax - b = 0$, has two imaginary roots, for all positive values of a.

PROP. V. *To determine whether an equation has equal roots.*

Let the proposed equation be

$$x^n + px^{n-1} + qx^{n-2} + \ldots\ldots + sx + t = 0 \ldots\ldots (1),$$

and let a be one root; then (*Prop.* I.) the equation (1) is divisible without remainder by $x - a$, and the quotient is

$$\left.\begin{array}{r|r|r} x^{n-1} + a & x^{n-2} + a^2 & x^{n-3} + \ldots\ldots + \ a^{n-1} \\ + p & + pa & \ldots\ldots + pa^{n-2} \\ & + \ q & \ldots\ldots + qa^{n-3} \\ & & \vdots \\ & & s \end{array}\right\} = 0 \ldots\ldots (2),$$

an equation containing all the remaining roots of (1). But if the equation (1) has two or more roots, each equal to a, then the substitution of a for x in equation (2) must necessarily verify that equation. Making this substitution, and recollecting that, as equation (1) has $n + 1$ terms, equation (2) has n terms, we have the several lines in (2) as below:—

$$\begin{array}{rl} a^{n-1} + a^{n-1} + a^{n-1} + \ldots\ldots \text{to } n \text{ terms} & = na^{n-1} \\ p\{a^{n-2} + a^{n-2} + a^{n-2} + \ldots\ldots \text{to } (n-1) \text{ terms}\} & = (n-1)pa^{n-2} \\ q\{a^{n-3} + a^{n-3} + a^{n-3} + \ldots\ldots \text{to } (n-2) \text{ terms}\} & = (n-2)qa^{n-3} \\ \vdots & \quad \vdots \\ s & \quad s \end{array}$$

$$\therefore\ na^{n-1} + (n-1)pa^{n-2} + (n-2)qa^{n-3} + \ldots\ldots + s = 0 \ldots\ldots (3).$$

But this is the same result as is furnished by writing a for x in the equation

$$nx^{n-1} + (n-1)px^{n-2} + (n-2)qx^{n-3} + \ldots\ldots + s = 0 \ldots\ldots (4)\cdot$$

Hence a is a root of equation (4), because by (3) it is verified by substituting in it a for x. We see then that, if equation (1) has two roots, each equal to a, equation (4) has one root equal to a, or in other words, $(x - a)^2$ is a factor of (1) and $x - a$ is a factor of (4); hence $x - a$ is a common measure of (1) and (4). The equation (4) is a very important one, and is derived from (1) by the following simple

Rule. Multiply each term of eq. (1) by the index of x in the same term, and diminish that index by unity.*

If then an equation has equal roots, they may be found by forming, from the given equation, the derived equation by the preceding rule; then if $x - a$ is the greatest common measure of these two equations, the given equation has two roots, each equal to a; and if $(x - a)^m$ is the greatest common measure, the proposed equation has $m + 1$ roots, each equal to a, and so on.

EXAMPLES.

1. The equation $x^4 - 7x^3 + 23x^2 - 40x + 28 = 0$, has equal roots; find them.

The derived equation $4x^3 - 21x^2 + 46x - 40 = 0$, contains one or more of the equal roots, and the greatest common measure of this and the given equation is $x - 2$; hence the proposed equation has two roots, each equal to 2. The equation may then be depressed to a quadratic, by *cor.* to *Prop.* II., and the other roots found.

2. Find whether the equation $x^3 - 2x^2 - 15x + 36 = 0$, has equal roots; and if so, determine them. Ans. 3, 3, -4.

3. Solve the equation $x^4 - 2x^3 - 3x^2 + 4x + 4 = 0$.

Ans. 2, 2, -1, -1.

4. Find the roots of the equation $x^6 - 3x^4 - 3x^3 + 10x^2 + 12x - 24 = 0$.

Ans. 2, 2, 2, $\dfrac{-3 \pm \sqrt{-3}}{2}$.

Prop. VI. *To transform an equation into others whose roots shall have certain relations to those of the given equation.*

The most useful transformations of an equation are the following, in all of which x is the unknown quantity in the given equation, and y that in the transformed equation.

---

* The last term t in (1) is equal to tx^0, which multiplied by 0 vanishes, and therefore it does not appear in (4).

(1) Let $x = y + r$, then the substitution of $y + r$ for x in the given equation will transform it into another, in which the roots are *less* or *greater* by r than those of the given equation, according as r is positive or negative, for $y = x - r$.

If the equation be of the nth degree, and p be the coefficient of the second term; then if $r = \frac{p}{n}$, the second term in the transformed equation will be wanting.

(2) Let $x = -y$, then the roots of the equation in y will be the same as those of the equation in x, but will have contrary signs, for $y = -x$.

(3) Let $x = \frac{1}{y}$, then the roots of the equation in y will be the reciprocals of the roots of the equation in x, for $y = \frac{1}{x}$.

(4) Let $x = \frac{y}{m}$, then the roots of the equation in y will be m times the roots of the equation in x, for $y = mx$.

(5) Let $x = \sqrt{y}$, then the roots of the equation in y, freed from radicals, will be the squares of the roots of the equation in x; for $y = x^2$.

The mode of effecting these substitutions is obvious enough, but the first transformation, being a most important one, will require special illustration.

Let $x^4 + px^3 + qx^2 + sx + t = 0$ be the given equation, and let it be required to transform it into another whose roots shall be less than those of the given equation by r. Now substituting $y + r$ for x gives

$$(y + r)^4 + p(y + r)^3 + q(y + r)^2 + s(y + r) + t = 0, \text{ or}$$

$$\left.\begin{array}{l|l|l|l} y^4 + 4r & y^3 + 6r^2 & y^2 + 4r^3 & y + r^4 \\ \quad + p & \quad + 3pr & \quad + 3pr^2 & \quad + pr^3 \\ & \quad + q & \quad + 2qr & \quad + qr^2 \\ & & \quad + s & \quad + sr \\ & & & \quad + t \end{array}\right\} = 0 \text{ .... } (1)$$

which is the transformed equation required; but if the division of $x^4 + px^3 + qx^2 + sx + t$ by $x-r$ be actually performed, we should have for quotient $x^3 + (r+p)x^2 + (r^2+pr+q)x + r^3 + pr^2 + qr + s$, and for remainder, an expression *identical* with the last term of eq. (1). Again, if this new quotient be divided by $x-r$, we should have for quotient $x^2 + (2r + p)x + (3r^2 + 2pr + q)$, and for remainder an expression *identical* with the coefficient of y in eq. (1). If the last quotient be divided by $x-r$, the remainder will be *identical* with the coefficient of y^2 in (1), and the quotient will be $x + (3r+p)$, which being divided by $x-r$, will evidently give the coefficient of y^3 in (1). This leads us to a beautiful and systematic process for effecting the transformation in equation (1), as will be seen in the following

EXAMPLES.

1. Find the equations whose roots shall be less by 2, and greater by 3, than those of the equation $x^4 - 4x^3 + 7x^2 + 10x - 20 = 0$.

Here we must divide the given equation by $x-2$ and $x+3$; and the quotients by the same divisors, as directed in the preceding remarks. Employing the synthetic mode of division (see pp. 24 and 25), the processes will be as follow:—

$$\begin{array}{l}
1-4+7+10-20(+2 \\
\quad +2-4+6+32 \\
\hline
1-2+3+16+12 \\
\quad +2+0+6 \\
\hline
1+0+3+22 \\
\quad +2+4 \\
\hline
1+2+7 \\
\quad +2 \\
\hline
1+4+7+22+12
\end{array}
\qquad\qquad
\begin{array}{l}
1-4+7+10-20(-3 \\
\quad -3+21-84+222 \\
\hline
1-7+28-74+202 \\
\quad -3+30-174 \\
\hline
1-10+58-248 \\
\quad -3+39 \\
\hline
1-13+97 \\
\quad -3 \\
\hline
1-16+97-248+202.
\end{array}$$

Hence the equations are respectively $y^4 + 4y^3 + 7y^2 + 22y + 12 = 0$, and $y^4 - 16y^3 + 97y^2 - 248y + 202 = 0$.

2. Transform the equation $x^4 - 8x^3 + 12x^2 - 5x + 4 = 0$ into another which shall want the second term.

By *Prop.* III., the sum of the roots of this equation is equal to 8, the coefficient of the second term; therefore we must *diminish* the sum of the roots by 8, or each root by 2, since there are four roots; hence we see that the taking away of the second term of an equation, is only a particular case of the first transformation in this proposition. The process is as follows:

$$\begin{array}{rrrrrl}
1 & -\ 8 & +\ 12 & -\ 5 & +\ 4 & (\ 2 \\
 & +\ 2 & -\ 12 & +\ 0 & -\ 10 & \\
\cline{1-5}
1 & -\ 6 & +\ 0 & -\ 5 & -\ 6 & \\
 & +\ 2 & -\ 8 & -\ 16 & & \\
\cline{1-4}
1 & -\ 4 & -\ 8 & -\ 21 & & \\
 & +\ 2 & -\ 4 & & & \\
\cline{1-3}
1 & -\ 2 & -\ 12 & & & \\
 & +\ 2 & & & & \\
\cline{1-5}
1 & +\ 0 & -\ 12 & -\ 21 & -\ 6 &
\end{array}$$

Hence the transformed equation is $y^4 - 12y^2 - 21y - 6 = 0$.

3. Find the least integer number that is greater than the greatest positive root of the equation $x^5 - 3x^4 - 11x^3 + 27x^2 + 10x - 25 = 0$.

Here we must diminish the roots successively by unity, till all the variations are lost, and the least number of units that does this is greater than the greatest positive root of the equation; for by *Prop.* IV., all the roots of the transformed equation will be negative. To save room, however, we shall first diminish the roots by 4.

$$\begin{array}{rrrrrrl}
1 & -\ 3 & -\ 11 & +\ 27 & +\ 10 & -\ 25 & (\ 4 \\
 & +\ 4 & +\ 4 & -\ 28 & -\ 4 & +\ 24 & \\
\cline{1-6}
1 & +\ 1 & -\ 7 & -\ 1 & +\ 6 & -\ 1 & \\
 & +\ 4 & +\ 20 & +\ 52 & +\ 204 & & \\
\cline{1-5}
1 & +\ 5 & +\ 13 & +\ 51 & +\ 210 & & \\
 & +\ 4 & +\ 36 & +\ 196 & & & \\
\cline{1-4}
1 & +\ 9 & +\ 49 & +\ 247 & & & \\
 & +\ 4 & +\ 52 & & & & \\
\cline{1-3}
1 & +\ 13 & +\ 101 & & & & \\
 & +\ 4 & & & & & \\
\cline{1-6}
1 & +\ 17 & +\ 101 & +\ 247 & +\ 201 & -\ 1 &
\end{array}$$

Hence it is obvious that by reducing the roots of this last equation, viz., $y^5 + 17y^4 + 101y^3 + 247y^2 + 201y - 1 = 0$, by unity, all the variations would be lost, and therefore 5 is greater than the greatest positive root of the equation. The greatest root is very little greater than 4, for the last term of the transformed equation, being -1, shews that a number a little greater than 4 would verify the proposed equation.

4. Diminish the roots of the equation $x^5 - 4x^3 + 6x^2 - 8 = 0$ by 1.

Ans. $y^5 + 5y^4 + 6y^3 + 4y^2 + 5y - 5 = 0$.

5. Find two equations, such that the roots of the one shall be less by 1·5, and the roots of the other greater by 2·5, than those of the equation $x^3 - 4x^2 + 6x - 8 = 0$.

$$\text{Ans.}\begin{cases} y^3 + \cdot 5y^2 + \cdot 75y - 4\cdot 625 = 0. \\ y^3 - 11\cdot 5y^2 + 44\cdot 75y - 63\cdot 625 = 0. \end{cases}$$

6. Find the equation whose roots are the reciprocals of those of the equation $x^4 - 3x^3 + 4x^2 + 6x - 12 = 0$.

Ans. $12y^4 - 6y^3 - 4y^2 + 3y - 1 = 0$.

7. Transform the equation $2x^4 + 3x^3 + 4x^2 - 5x - 6 = 0$ into another whose roots are the same but with changed signs, and also into one whose roots are three times those of the given equation.

$$\text{Ans.}\begin{cases} 2y^4 - 3y^3 + 4y^2 + 5y - 6 = 0. \\ 2y^4 + 9y^3 + 36y^2 - 135y - 486 = 0. \end{cases}$$

8. Find two consecutive numbers, the one less and the other greater than the greatest positive root of the equation $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$.

Ans. 5 and 6.

PROP. VII. *If* a_1, a_2, a_3...... *denote the real roots of an equation* $fx = 0$, *where* a_1 *is the greatest,* a_2 *the next in order of magnitude, and so on; then if a series of numbers* p_1, p_2, p_3,...... *(in which* p_1 *is greater than* a_1, p_2 *a number less than* a_1 *but greater than* a_2, *and so on) be substituted for* x *in the polynomial* fx, *the results will be alternately positive and negative.*

Since (*Prop.* II.), $fx = (x-a_1)(x-a_2)(x-a_3)(x-a_4)$ .... ; where the quadratic factors involving imaginary roots are omitted, since they are always positive for all values of x (*Prop.* III., p. 201), therefore by substituting for x the proposed series of values $p_1, p_2, p_3 \ldots\ldots$, we have, by hypothesis,

$$\begin{array}{l}
(p_1-a_1)(p_1-a_2)(p_1-a_3)(p_1-a_4)\ldots\ldots = +.+.+.+\ldots\ldots = + \\
(p_2-a_1)(p_2-a_2)(p_2-a_3)(p_2-a_4)\ldots\ldots = -.+.+.+\ldots\ldots = - \\
(p_3-a_1)(p_3-a_2)(p_3-a_3)(p_3-a_4)\ldots\ldots = -.-.+.+\ldots\ldots = + \\
(p_4-a_1)(p_4-a_2)(p_4-a_3)(p_4-a_4)\ldots\ldots = -.-.-.+\ldots\ldots = -, \\
\quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots
\end{array}$$

Cor. If two numbers be substituted for x in any equation $fx = 0$, and give results with contrary signs, then between these numbers there must be *one* root, or an odd number of roots ; but if the results have the same sign, then between these numbers there must be *no* root, or an even number of roots.

BUDAN'S CRITERION OF IMAGINARY ROOTS.

If in the process for reducing the roots of an equation by a number r, there be p variations of sign *lost*, and if in reducing the roots of the reciprocal equation by $\frac{1}{r}$, there be q variations *left;* then will there be in the interval 0 to r at least $p - q$ imaginary roots.

For suppose that the proposed equation has three variations of sign, indicating, if the roots are real, three positive roots, and let 1, 2, 5 be the roots. Reducing the roots by 6, the three positive roots will become negative, since 6 is greater than each of them, and the signs of the terms of the transformed equation will be all positive, and *three* variations are *lost.* Again, the roots of the reciprocal equation are 1, $\frac{1}{2}$, $\frac{1}{5}$, which are all positive, and hence the equation whose roots are 1, $\frac{1}{2}$, $\frac{1}{5}$, will furnish *three* variations of sign ; but in reducing the roots of the reciprocal equation by $\frac{1}{6}$, the reciprocal of 6 the former reducing number, the roots are still positive, since $\frac{1}{6}$ is *less* than each of the roots ; hence *three* variations must be *left*, and the criterion indicates no imaginary roots, as it ought. But if p, the number of variations *lost* in the transformation of the equation by r, be not equal to, but greater than q, the number of variations *left* in the transformed reciprocal equation by $\frac{1}{r}$, there will be a contradiction respecting the character (positive or negative) of a number of the roots equal to $p-q$, and these roots must therefore be imaginary.

Note. This is a most valuable criterion, and fully equal to the task, when employed in a skilful manner, of detecting the number of imaginary roots in any equation.

STURM'S CRITERION OF REAL ROOTS.

Let $\mathrm{x} = ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots + sx + t = 0$, be an equation which has not equal roots, and let $\mathrm{x}_1 = nax^{n-1} + (n-1)bx^{n-2} + (n-2)cx^{n-3} + \ldots\ldots + s$ be the function of x derived from the former as in *Prop.* V., by multiplying each term of the equation $\mathrm{x} = 0$, by the index of x in the same term, and diminishing that index by unity. To the expressions x and x_1 apply the process of finding the greatest common measure of two quantities (p. 27), changing, however, the signs of all the remainders. Call the first remainder thus modified x_2, the second remainder thus changed x_3, and so on, till a remainder is found which is independent of x, and let the sign of this remainder be also changed. In these several expressions $\mathrm{x}, \mathrm{x}_1, \mathrm{x}_2, \ldots\ldots$ substitute for x a number p, and count the number of variations of sign in the results. In the same expressions substitute for x a number q, greater than p, and count the number of variations of sign in these results. Then if the substitution of p give h variations, and the substitution of q give k variations, the *exact* number of real roots between the numbers p and q will be equal to $h-k$.

$$\begin{array}{l} \mathrm{x}_1\ )\ \mathrm{x}\ \ (\ \mathrm{Q}_1 \\ \quad\ \mathrm{x}_1\mathrm{Q}_1 \\ \hline \mathrm{x} - \mathrm{x}_1\mathrm{Q}_1 = -\,\mathrm{x}_2 \\ \text{or } -\mathrm{x} + \mathrm{x}_1\mathrm{Q}_1 = \mathrm{x}_2 \end{array}$$

$$\begin{array}{l} \mathrm{x}_2\ )\ \mathrm{x}_1\ \ (\ \mathrm{Q}_2 \\ \quad\ \mathrm{x}_2\mathrm{Q}_2 \\ \hline -\mathrm{x}_1 + \mathrm{x}_2\mathrm{Q}_2 = \mathrm{x}_3 \\ \quad\ \&c. \end{array}$$

To prove this theorem we may proceed as follows :—

(1) By the preceding marginal process we have

$$\mathrm{x} = \mathrm{x}_1\mathrm{Q}_1 - \mathrm{x}_2,\ \mathrm{x}_1 = \mathrm{x}_2\mathrm{Q}_2 - \mathrm{x}_3, \ldots\ldots\ \mathrm{x}_{n-2} = \mathrm{x}_{n-1}\mathrm{Q}_{n-1} - \mathrm{x}_n.$$

Now if any two consecutive functions of the series $\mathrm{x}, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \ldots \mathrm{x}_n$, can vanish simultaneously by the substitution of the same value of x, we should have all the succeeding functions equal to 0, which is absurd, since the last remainder x_n is independent of x. Thus, if x_1 and x_2 were each equal to 0, then $x_3 = 0$, and so on.

(2) Since $x_2 = x_3Q_3 - x_4$, if $x_3 = 0$, then $x_2 = -x_4$; therefore if one of the intermediate functions x_1, x_2, x_3...... vanish for some particular value of x, the two adjacent functions will have opposite signs for the same value of x, and therefore the loss of a variation of sign cannot occur amongst these intermediate functions on this account.

(3) From (1) and (2) we see that the number of variations can only be affected by those values of x which immediately precede and follow a root of the equation $x = 0$, the sign of the remainder being invariable. Hence when x passes a root of the equation, one variation of sign will be lost (*Prop.* VII.), and in a similar manner another variation will be lost when x passes the next higher root, for although these variations are successively transmitted amongst the signs by the changes of x_1, x_2, x_3,...... yet we have seen (1, 2) that such transmission cannot affect the number of variations, unless x passes over a root of the equation $x = 0$. Hence the truth of the theorem is established.

Note. The limits of all the roots are $-\infty$ and $+\infty$, the limits of the negative roots are 0 and $-\infty$, and the limits of the positive ones are 0 and $+\infty$.

HORNER'S METHOD OF RESOLVING NUMERICAL EQUATIONS.

When the initial figures of the incommensurable roots of an equation have been ascertained by one of the preceding processes, their values may be obtained to any degree of accuracy by the method of approximation devised by the late ingenious Mr W. G. HORNER, of Bath. This process of approximation consists in the diminishing a root by its first figure (*Prop.* VI.); and repeating the process on each transformed equation, till the root be obtained to any degree of accuracy required. The application of these criteria, and the illustration of HORNER'S method of approximation, will be understood from the examples subjoined.

Ex. Analyse completely the equation $x^4 - 5x^3 + 3x^2 + 7x - 13 = 0$, and find the approximate values of the real roots to eight places of decimals.

By Sturm's Criterion.

Here the given and derived functions are found to be

$$\begin{aligned} \mathrm{X} &= x^4 - 5x^3 + 3x^2 + 7x - 13 \\ \mathrm{X}_1 &= 4x^3 - 15x^2 + 6x + 7 \\ \mathrm{X}_2 &= 51x^2 - 114x + 173 \\ \mathrm{X}_3 &= 1144x - 1493 \\ \mathrm{X}_4 &= - \end{aligned}$$

Substituting in each of these,

for $x = -\infty$, the signs are $+ \; - \; + \; - \; -$ three variations

$x = 0$ . . . . $- \; + \; + \; - \; -$ two variations

$x = +\infty$ . . . . . . $+ \; + \; + \; + \; -$ one variation,

consequently the proposed equation has only two real roots, one negative and one positive; and to determine the initial figures of these roots, we must narrow the limits. Thus the signs of $\mathrm{X}, \mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4$, are

for $x = 0$, $- + + - -$, 2 var. for $x = 0$, $- + + - -$, 2 var.

$x = -1$, $- - + - -$, 2 var. $x = 4$, $- + + + -$, 2 var.

$x = -2$, $+ - + - -$, 3 var. $x = 5$, $+ + + + -$, 1 var.

consequently the negative root is between -2 and -1, and the positive root between 4 and 5.

By Budan's Criterion.

If the roots of the proposed equation are real, there will be three positive roots and one negative root (*Prop.* IV.); hence reducing by unity, we have the following operation :—

$$\begin{array}{l}
1 - 5 + 3 + 7 - 13(1 \\
\quad - 4 - 1 + 6 - 7 \\
\quad - 3 - 4 + 2 \\
\quad - 2 - 6 \\
1 - 1 - 6 + 2 - 7(1 \\
\quad + 0 - 6 - 4 - 11 \\
\quad + 1 - 5 - 9 \\
\quad + 2 - 3 \\
1 + 3 - 3 - 9 - 11 \quad \text{2 var. lost.}
\end{array}$$

Reciprocal equation.

$$\begin{array}{l}
7 - 2 + 6 + 1 - 1(1 \\
\quad + 5 + 11 + 12 + 11 \\
\quad \text{no variations left.}
\end{array}$$

Since two variations are *lost* in the last transformation of the direct equation, and no variations are *left* in the transformed reciprocal equation, the proposed equation has two imaginary roots.

To find the situation of the two real roots, one of which is positive, and the other negative, we must continue the process of reduction till the variation is lost: thus

$$\begin{array}{l}
1 + 3 - 3 - 9 - 11(1 \\
\quad + 4 + 1 - 8 - 19 \\
\quad + 5 + 6 - 2 \\
\quad + 6 + 12 \\
1 + 7 + 12 - 2 - 19
\end{array}
\qquad
\begin{array}{l}
1 + 7 + 12 - 2 - 19(1 \\
\quad + 8 + 20 + 18 - 1 \\
\quad + 9 + 29 + 47 \\
\quad + 10 + 39 \\
1 + 11 + 39 + 47 - 1
\end{array}$$

The next reduction by unity will obviously make all the signs positive, and therefore the positive root lies between 4 and 5. To find the situation of the negative root, change the signs of the alternate terms, (Prop. VI. 2), and proceed as before.

$$\begin{array}{l}
1 + 5 + 3 - 7 - 13(1 \\
\quad + 6 + 9 + 2 - 11 \\
\quad + 7 + 16 + 18 \\
\quad + 8 + 24 \\
1 + 9 + 24 + 18 - 11
\end{array}
\qquad
\begin{array}{l}
1 + 9 + 24 + 18 - 11(1 \\
\quad + 10 + 34 + 52 + 42 \\
\quad \text{signs all } +
\end{array}$$

Hence the negative root is situated between — 2 and — 1, as before determined by Sturm's criterion. Both criterions are unfailing; and in practice, Budan's has the advantage of Sturm's, as it requires a much less amount of labour.

We shall now find the values of these roots by Horner's method.*

<table>
<tr><td>1 — 5</td><td>+ 3</td><td>+ 7</td><td>— 13</td><td>(4·020911590</td></tr>
<tr><td>+ 4</td><td>— 4</td><td>— 4</td><td>+ 12</td><td></td></tr>
<tr><td>— 1</td><td>— 1</td><td>+ 3</td><td>— 1·00000000</td><td></td></tr>
<tr><td>+ 4</td><td>+ 12</td><td>+ 44</td><td>+ ·95568816</td><td></td></tr>
<tr><td>3</td><td>11</td><td>47000000</td><td>— ·04431184</td><td></td></tr>
<tr><td>4</td><td>28</td><td>784408</td><td>+ ·04374803</td><td></td></tr>
<tr><td>7</td><td>390000</td><td>47784408</td><td>— ·00056381</td><td></td></tr>
<tr><td>4</td><td>2204</td><td>788824</td><td>48645</td><td></td></tr>
<tr><td>1102</td><td>392204</td><td>485732|3|2</td><td>— 7736</td><td></td></tr>
<tr><td>2</td><td>2208</td><td>356|9</td><td>4864</td><td></td></tr>
<tr><td>1104</td><td>394412</td><td>486089|2</td><td>— 2872</td><td></td></tr>
<tr><td>2</td><td>2212</td><td>356|9</td><td>2432</td><td></td></tr>
<tr><td>1106</td><td>|39|66|24</td><td>48644|6</td><td>— 440</td><td></td></tr>
<tr><td>2</td><td></td><td>|4</td><td>438</td><td></td></tr>
<tr><td>|· · 1|108</td><td></td><td>4|8|6|4|5</td><td>— 2</td><td></td></tr>
</table>

hence the positive root is =4·020911590 .....

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* The process here employed is precisely similar to that used in the modern method of extracting the cube root of a number; and those who are familiar with the latter process, and the contractions for obtaining the last few figures of the root, will find no difficulty in determining the roots of equations to any extent.

For the negative root, change the signs of the alternate terms of the equation, and then proceed as above, prefixing the negative sign to the result.

1	+ 5	+ 3	— 7	— 13	(1·384435487
	+ 1	+ 6	+ 9	+ 2	
	6	9	2	— 11·0000	
	1	7	16	78111	
	7	16	18000	— 318890000	
	1	8	8037	300919936	
	8	2400	26037	— 17970064	
	1	279	8901	16199585	
	93	2679	34938000	— 1770479	
	3	288	2676992	1626156	
	96	2967	37614992	— 144323	
	3	297	2743296	122007	
	99	326400	4035828|8	— 22316	
	3	8224	14067|4	20335	
	1028	334624	4049896|2	— 1981	
	8	8288	14084|	1627	
	1036	342912	406398|0	— 354	
	8	8352	141|0	325	
	1044	3512|64	406539	29	
	8	4|21	141		
| . .	1|052	3516|85	40668|0		
		4|2	1|		
		3521|	40669		
		4|	1		
		|35|25	4|0|6|7|0		

Consequently the negative root is = — 1·384435487.

EXAMPLES FOR PRACTICE.

Analyse completely the following equations, and find their real roots to seven places of decimals.

(1) $x^3 - 7x + 7 = 0.$ (2) $x^3 + 11x^2 - 102x + 181 = 0.$

(3) $x^4 + x^3 - x^2 - 2x + 4 = 0.$ (4) $x^4 + x^3 + x^2 + 3x - 100 = 0.$

(5) $x^4 + x^3 + 4x^2 - 4x + 1 = 0.$ (6) $x^4 - x^3 - 4x^2 + 4x + 1 = 0.$

(7) $x^3 - 2x - 5 = 0.$ (8) $x^3 + 10x^2 + 5x - 2600 = 0.$

(9) $x^5 - 10x^3 + 6x + 1 = 0.$ (10) $x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0.$

(11) $x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0.$

(12) $2x^4 - 1400x^3 - 5x^2 - 11615x + 29145 = 0.$

ANSWERS.

(1) $x = 1{\cdot}356895867$, $x = 1{\cdot}692021471$, $x = -3{\cdot}048917339$.

(2) $x = 3{\cdot}21312775$, $x = 3{\cdot}22952121$, $x = -17{\cdot}44264896$.

(3) All the four roots are imaginary.

(4) $x = 2{\cdot}802851218$, $x = -3{\cdot}433577863$, and the others imaginary.

(5) All the roots are imaginary.

(6) $x = 1{\cdot}33826121$, $x = 1{\cdot}82709092$, $x = -{\cdot}20905692$, $x = -1{\cdot}95629520$.

(7) $x = 2{\cdot}09455148154$, and the other roots are imaginary.

(8) $x = 11{\cdot}00679934$, and the other roots are imaginary.

(9) $x = {\cdot}8795087$, $x = 3{\cdot}0530581$, $x = -{\cdot}1756748$, $x = -{\cdot}6915763$, $x = -3{\cdot}0653158$.

(10) $x = 1{\cdot}05910900346$, and the other roots are imaginary.

(11) $x = {\cdot}3509870$, $x = 12{\cdot}7564418$, $x = 32{\cdot}0602909$, $x = 34{\cdot}8322803$.

(12) $x = 1{\cdot}80309288$, $x = 700{\cdot}015380387$, the others are imaginary.

RECURRING AND BINOMIAL EQUATIONS.

A *recurring* or *reciprocal equation* is one in which the coefficients recur in a *reverse* order, either with the same or opposite signs. Thus

$$x^5 + px^4 + qx^3 + qx^2 + px + 1 = 0 \ldots\ldots(1)$$

$$x^4 + px^3 - px - 1 = 0 \ldots\ldots(2)$$

$$x^n - 1 = 0 \ldots\ldots(3)$$

are all recurring or reciprocal equations. In the first of these equations write $\frac{1}{y}$ for x, and the resulting equation in y will have the same coefficients as (1), and its roots are the reciprocals of the roots of (1), for $y = \frac{1}{x}$. The roots of (1) must therefore be of the form

$$r_1, \frac{1}{r_1}, r_2, \frac{1}{r_2}, \text{ and } -1.$$

Equation (1) may be written thus :—

$$(x^5 + 1) + px(x^3 + 1) + qx^2(x + 1) = 0,$$

which is divisible by $x + 1$, and therefore one root is $= -1$. The quotient, after dividing by $x + 1$, put $= 0$, gives the equation

$$x^4 + (p - 1)x^3 + (q - p + 1)x^2 + (p - 1)x + 1 = 0,$$

a recurring equation, and dividing by x^2, and arranging, we get

$$x^2 + x^{-2} + (p - 1)(x + x^{-1}) = -(q - p + 1)$$

$$\text{or } (x + x^{-1})^2 + (p - 1)(x + x^{-1}) = -(q - p - 1),$$

by adding 2 to both members. Solving this quadratic, gives the values of $x + x^{-1}$, and thence, finally, the values of x may be obtained. Eq. (2) is solved in a similar manner, and the binomial equation $x^5 - 1 = 0$ may be resolved by dividing the first member by $x - 1$, which gives $x^4 + x^3 + x^2 + x + 1 = 0$, for the equation containing the four other roots. The roots of $x^5 + 1 = 0$ are the same as those of $x^5 - 1 = 0$, with their signs changed. The three roots of $x^3 - 1 = 0$ are 1, $\frac{1}{2}(-1 + \sqrt{-3})$, $\frac{1}{2}(-1 - \sqrt{-3})$, and the three roots of $x^3 + 1 = 0$ are -1, $\frac{1}{2}(1 - \sqrt{-3})$ and $\frac{1}{2}(1 + \sqrt{-3})$.

INDETERMINATE ANALYSIS.

In the common rules of algebra, such questions are usually proposed as require some certain or definite answer; in which case it is necessary that there should be as many independent equations, expressing the conditions of the question, as there are unknown quantities to be determined; or otherwise the problem would not be limited.

But in other branches of the science, questions frequently arise that involve a greater number of unknown quantities than there are equations to determine them; in which instances they are called indeterminate or unlimited problems; being such as usually admit of an unlimited number of solutions; although, when the question is proposed in integers, and the answers are required only in whole positive numbers, they are, in some cases, confined within certain limits, and in others the problem may become impossible.

PROBLEM I. To find the positive integral values of x and y in the equation

$$ax - by = c;$$

where a, b, c are given whole numbers, positive or negative, and where a and b have no common divisor, except when c is also divisible by it, and then the equation might be reduced.

Rule. Let a be the coefficient which is less than b; then the equation gives

$$x = \frac{by + c}{a} = my \pm n \pm \frac{dy \pm e}{a};$$

where my and n are the *nearest* quotients, whether in defect or excess, arising from dividing by and c by a; hence d and e are each less than a. Let the last fraction, which must be a whole number, be put $= v$, then $dy \pm e = av$, and as before

$$y = \frac{av \mp e}{d} = m'v \pm n' \pm \frac{fv \pm g}{d};$$

where f and g are each less than d. Calling the last fraction v_1, and repeating the process, the coefficient of v_1 would be found to be still farther diminished. Thus, by repeated operations, we at length arrive at a fractional expression, in which the coefficient of the subsidiary unknown quantity v_n is unity. Put this fraction $= p$, and we get an expression of the form

$$v_n = hp \pm k.$$

By reversing our steps we get in succession the values of the subsidiary unknown quantities which have been employed, and finally those of x and y.*

Note. Any indeterminate equation of the form

$$ax - by = \pm c,$$

in which a and b are prime to each other, is always possible, and will admit of an unlimited number of answers in whole numbers.

But if the proposed equation be of the form

$$ax + by = c,$$

the number of answers will always be limited; and in some cases the question is impossible; both of which circumstances may be readily discovered, from the mode of solution above given.†

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* This rule is founded on the obvious principle, that the sum, difference, or product of any two whole numbers, is a whole number, and that, if a number divides the whole of any other number and a part of it, it will also divide the remaining part.

† That the coefficients a and b, when these two formulæ are possible, should have no common divisor, which is not at the same time a divisor of c, is evident; for if $a = md$, and $b = me$, we shall have $ax \pm by = m(dx \pm ey) = c$; and consequently $dx \pm ey = c \div m$.

But d, e, x, and y, being supposed to be whole numbers, $c \div m$ must also be a whole number, which it cannot be, except when m is a divisor of c.

Hence, if it were required to pay L.100 in guineas and moidores only, the question would be impossible; since, in the equation $21x + 27y = 2000$, which represents the conditions of the problem, the coefficients 21 and 27, are each divisible by 3, whilst the absolute term 2000 is not divisible by it. See Vol. II. of BONNYCASTLE'S *Treatise on Algebra*, for the method of resolving questions of this kind, by means of *Continued Fractions*.

EXAMPLES.

1. Given $19x - 14y = 11$, to find the positive integral values of x and y.

$$\text{Here } y = \frac{19x - 11}{14} = x - 1 + \frac{5x + 3}{14};$$

let the last fraction $= v$, then $5x + 3 = 14v$; consequently

$$x = \frac{14v - 3}{5} = 3v - 1 - \frac{v - 2}{5}.$$

Put the last fraction $= p$, then $v - 2 = 5p$, or $v = 5p + 2$; hence

$$x = 3v - 1 - p = 14p + 5,$$

$$y = x - 1 + v = 19p + 6.$$

$$\text{Let } p = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ etc.;$$

$$\text{then } x = 5,\ 19,\ 33,\ 47,\ 61,\ 75,\ etc.,$$

$$y = 6,\ 25,\ 44,\ 63,\ 82,\ 101,\ etc.$$

In this problem, the number of answers is unlimited.

2. Given $21x + 17y = 2000$, to find all the integral values of x and y.

$$\text{Here } y = \frac{2000 - 21x}{17} = 118 - x - \frac{4x + 6}{17};$$

let the last fraction $= v$, then $4x + 6 = 17v$; hence

$$x = \frac{17v - 6}{4} = 4v - 1 + \frac{v - 2}{4}.$$

Put the last fraction $= p$; then $v = 4p + 2$, and hence

$$x = 4v - 1 + p = 17p + 7,$$

$$y = 118 - x - v = 109 - 21p.$$

$$\text{Let } p = 0,\ 1,\ 2,\ 3,\ 4,\ 5,$$

$$\text{then } x = 7,\ 24,\ 41,\ 58,\ 75,\ 92,$$

$$y = 109,\ 88,\ 67,\ 46,\ 25,\ 4.$$

In this question the number of answers is limited to six.

If there be three unknown quantities, and only two equations for determining them, as

$$ax + by + cz = d, \text{ and } ex + fy + gz = h,$$

eliminate one of the unknown quantities in the usual manner, and find the values of the other two from the resulting equation.

But if there be three unknown quantities, and only one equation by which they are to be found, as $ax + by + cz = d$, we must first find the limit of the quantity that has the greatest coefficient, and then ascertain the different values of the other two, by separate substitutions of the several values of the former, from 1 up to the limit, as in the following example.

3. Given $3x + 5y + 7z = 100$, to find all the positive integral values of x, y, and z.

Here $x = 1$, and $y = 1$ are the least values of these two quantities, and

$$z = \frac{100-3-5}{7} = 13\tfrac{1}{7};$$

therefore the limit of the value of z is 13. Supposing then that z is known, we have

$$x = \frac{100-7z-5y}{3} = 33 - 2z - 2y + \frac{y-z+1}{3}.$$

Let the last fraction $= p$, then $y - z + 1 = 3p$, and hence

$$y = 3p + z - 1,$$
$$x = 33 - 2z - 2y + p = 35 - 4z - 5p.$$

By giving to z all values from 1 to its limit 13, and to p all values, positive or negative, which furnish integral and positive values of x and y, we get the entire number of solutions, as follows:

z	y	x
1	3	26
1	6	21
1	9	16
1	12	11
1	15	6
1	18	1
2	1	27
2	4	22
2	7	17
2	10	12
2	13	7
2	16	2
3	2	23
3	5	18
3	8	13
3	11	8
3	14	3
4	3	19
4	6	14
4	9	9
4	12	4
5	1	20
5	4	15
5	7	10
5	10	5
6	2	16
6	5	11
6	8	6
6	11	1
7	3	12
7	6	7
7	9	2
8	1	13
8	4	8
8	7	3
9	2	9
9	5	4
10	3	5
11	1	6
11	4	1
12	2	2

The number of solutions is limited to 41.

EXAMPLES FOR EXERCISE.

1. Given $2x + 3y = 25$ to determine x and y in whole numbers.

Ans. $x = 11, 8, 5, 2$
$y = 1, 3, 5, 7.$

2. Given $3x = 8y - 16$, to find the least values of x and y.

Ans. $x = 8$, $y = 5$.

3. Given $14x = 5y + 7$, to find the least values of x and y.

Ans. $x = 3$, $y = 7$.

4. Given $27x = 1600 - 16y$, to find the least values of x and y.

Ans. $x = 48$, $y = 19$.

5. It is required to divide 100 into two such parts, that one of them may be divisible by 7, and the other by 11.

Ans. The only parts are 56 and 44.

6. Given $9x + 13y = 2000$, to find the greatest value of x, and the least value of y. Ans. $x = 215$, $y = 5$.

7. Given $11x + 5y = 254$, to find all the possible values of x and y.

Ans. $x = 19, 14, 9, 4$; $y = 9, 20, 31, 42$.

8. Given $x - 2y + z = 5$, and $2x + y - z = 7$, to find the values of x, y, and z.

$$\text{Ans.}\left\{\begin{array}{llllll} x = 5, & 6, & 7, & 8, & 9, & \ldots\ldots \\ y = 3, & 6, & 9, & 12, & 15, & \ldots\ldots \\ z = 6, & 11, & 16, & 21, & 26, & \ldots\ldots \end{array}\right.$$

9. Given $17x + 19y + 21z = 400$, to find all the answers in whole numbers which the question admits of.

Ans. 10 different answers.

10. Given $5x + 7y + 11z = 224$, to find all the possible values of x, y, and z, in whole positive numbers.

The number of answers is 59.

11. It is required to find in how many different ways it is possible to pay L.20 in half-guineas and half-crowns, without using any other sort of coin? Ans. In 7 different ways.

12. There were bought 100 animals for L.100; oxen at L.5 each, sheep at L.1 each, and fowls at 1s. each; how many were there of each kind? Ans. 19 oxen, 1 sheep, and 80 fowls.

13. I owe my friend a shilling, and have nothing about me but guineas, and he has nothing but louis-d'ors; how must I contrive to pay the debt, the louis being valued at 17s. apiece, and the guineas at 21s.?

Ans. I must give him 13 guineas, and he must give me 16 louis.

14. How many gallons of spirits, at 12s., 15s., and 18s. a gallon, must be taken to make a mixture of 1000 gallons, that shall be worth 17s. a gallon? Ans. $111\frac{1}{9}$ at 12s., $111\frac{1}{9}$ at 15s., $777\frac{7}{9}$ at 18s.

Problem II. To find a whole number, which, being divided by other given numbers, shall leave given remainders.

Rule. Let x denote the number required, a, b, c, *etc.* the given divisors, and f, g, h, *etc.* the given remainders.

Subtract each remainder from x, and divide the differences by a, b, c, *etc.*, and there will arise

$$\frac{x-f}{a},\ \frac{x-g}{b},\ \frac{x-h}{c},\ \textit{etc.} = \text{whole numbers.}$$

Put the first of these fractions $= p$, and substitute the value of x, as found in terms of p, from this equation, in the place of x in the second fraction.

Find the least value of p in this second fraction, by the last problem, which put $= r$, and substitute the value of x, as found in terms of r, in the place of x in the third fraction.

Find, in like manner, the least value of r, in this third fraction, which put $= s$, and substitute the value of x, as found in terms of s, in the fourth fraction, as before.

Proceed in the same way with the next following fraction, and so on, to the last; when the value of x, thus determined, will give the whole number required.

When only two fractions are to be found in whole numbers, the operation may be conducted as in the first of the following examples.

EXAMPLES FOR EXERCISE.

1. It is required to find the least whole number, which, being divided by 17, shall leave a remainder of 7, and, when divided by 26, shall leave a remainder of 13.

Let x denote the required number, then we have

$$\frac{x-7}{17} = v \text{ and } \frac{x-13}{26} = v_1.$$

From the former, $x = 17v + 7$, and from the latter $x = 26v_1 + 13$; hence

$$17v + 7 = 26v_1 + 13, \text{ or } 17v = 26v_1 + 6;$$

$$\therefore\ v = \frac{26v_1 + 6}{17} = 2v_1 - \frac{8v_1 - 6}{17} = 2v_1 - 2.\frac{4v_1 - 3}{17}.$$

Let the last fractional part $= v_2$; then $4v_1 = 17v_2 + 3$; hence

$$v_1 = \frac{17v_2 + 3}{4} = 4v_2 + 1 + \frac{v_2 - 1}{4}.$$

Put the last fraction $= p$; then $v_2 = 4p + 1$; consequently

$$v_1 = 4v_2 + 1 + p = 17p + 5,$$

$$x = 26v_1 + 13 = 442p + 143.$$

If $p = 0$, the least value of x is 143, a number which fulfils the conditions of the question, and others may be found at pleasure, by giving different values to p.

2. It is required to find the least whole number, which, being divided by 11, 19, and 29, shall leave the remainders 3, 5, and 10, respectively.
Ans. 4128.

3. To find a number, which, being divided by 6, shall leave the remainder 2, and when divided by 13, shall leave the remainder 3.
Ans. 68.

4. It is required to find a number, which, being divided by 7, shall leave 5 for a remainder, and if divided by 9, the remainder shall be 2.
Ans. 47.

5. It is required to find the least whole number, which, being divided by 39, shall leave the remainder 16, and when divided by 56, the remainder shall be 27. Ans. 1147.

6. It is required to find the least whole number, which, being divided by 7, 8, and 9, respectively, shall leave the remainders 5, 7, and 8
Ans. 215.

7. It is required to find the least whole number, which, being divided by each of the nine digits, 1, 2, 3, 4, 5, 6, 7, 8, 9, shall leave no remainders. Ans. 2520.

8. A person receiving a box of oranges, observed, that when he told them out by 2, 3, 4, 5, and 6 at a time, he had none remaining; but when he told them out by 7 at a time, there remained 5; how many oranges were there in the box, the number being the least possible? Ans. 180.

DIOPHANTINE ANALYSIS.

This branch of analysis takes its name from that of a Greek mathematician, Diophantus of Alexandria, in Egypt, who flourished in or about the fourth century after Christ. Its object is to find such values of quantities, that certain functions of them may become squares or cubes.

Prop. I. *To find such values of x, that the expression $ax^2 + bx + c$ may be a square.*

Case I. When $a = 0$, let $\sqrt{(bx + c)} = h$, then $x = \dfrac{h^2 - c}{b}$......(1).

Case II. When $c = 0$, let $\sqrt{(ax^2 + bx)} = hx$, then $x = \dfrac{b}{h^2 - a}$....(2).

Case III. When a is a square, m^2, let $\sqrt{(m^2x^2 + bx + c)} = mx - h$; then squaring, and reducing gives $x = \dfrac{h^2 - c}{b + 2mh}$....................(3).

Case IV. When c is a square as n^2, let $\sqrt{(ax^2 + bx + n^2)} = hx - n$; then squaring and reducing gives $x = \dfrac{b + 2nh}{h^2 - a}$..........................(4).

Case V. When $ax^2 + bx + c$ is composed of two factors, as $(mx + n)(px + q)$; let $\sqrt{(mx + n)(px + q)} = h(px + q)$; then squaring and dividing by $px + q$, we get $mx + n = h^2(px + q)$, and

$$x = -\frac{qh^2 - n}{ph^2 - m}\text{..............................(5).}$$

CASE VI. When $ax^2 + bx + c$ is the sum of a square, and the product of two factors, as $(mx + n)^2 + (px + q)(rx + s)$; then let

$$(mx+n)^2+(px+q)(rx+s)=\{(mx+n)+h(px+q)\}^2$$
$$= (mx + n)^2 + 2h(mx+n)(px + q) + h^2(px + q)^2;$$

hence $rx + s = 2h(mx + n) + h^2(px + q)$, and therefore

$$x = -\frac{qh^2 + 2nh - s}{ph^2 + 2mh - r} \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Scholium. If one value of x can be found that will render ax^2+bx+c a square, we can find other values in this manner. Let r be a value of x which renders $ax^2 + bx + c$ a square, and let $x = y + r$; then $ax^2 + bx + c$ becomes $ay^2 + (b + 2ar)y + ar^2 + br + c$; but by hypothesis, $ar^2 + br + c$ is a square, and hence by Case IV.,

$$y = \frac{b + 2ar + 2nh}{h^2 - a}, \text{ where } n = \sqrt{(ar^2 + br + c)};$$

$$\text{and } x = y + r = \frac{b + ar + 2nh + rh^2}{h^2 - a}. \ldots\ldots\ldots\ldots(7).$$

In all these cases, the value of h is arbitrary, and may be assumed at pleasure.

PROP. II. *To find such values of x, that the expression $ax^3 + bx^2 + cx + d$ may be a cube.*

CASE I. If a is a cube, as m^3, then let

$$m^3x^3 + bx^2 + cx + d = \left(mx + \frac{b}{3m^2}\right)^3 = m^3x^3 + bx^2 + \frac{b^2x}{3m^3} + \frac{b^3}{27m^6};$$

$$\therefore\ x = \frac{b^3 - 27dm^6}{9m^3(3cm^3 - b^2)} \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

CASE II. If d is a cube, as n^3, then let

$$ax^3 + bx^2 + cx + n^3 = \left(\frac{cx}{3n^2} + n\right)^3 = \frac{c^3x^3}{27n^6} + \frac{c^2x^2}{3n^3} + cx + n^3;$$

$$\therefore \quad x = \frac{9n^3(c^2 - 3bn^3)}{27an^6 - c^3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Case III. If a is a cube, m^3, and d is also a cube, n^3; then let

$$m^3x^3 + bx^2 + cx + n^3 = (mx + n)^3 = m^3x^3 + 3m^2nx^2 + 3mn^2x + n^3;$$

$$\therefore \quad x = \frac{3mn^2 - c}{b - 3m^2n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Note. If one value of x can be found, by trial or otherwise, then other values may be found as in the *Scholium* to *Prop.* I.

Prop. III. *To find such values of* x, *that the expression* $ax^4 + bx^3 + cx^2 + dx + e$ *may be a square.*

Case I. If a is a square as m^2, then let

$$m^2x^4 + bx^3 + cx^2 + dx + e = (mx^2 + \frac{b}{2m}x + h)^2$$

$$= m^2x^4 + bx^3 + \left(\frac{b^2}{4m^2} + 2mh\right)x^2 + \frac{bh}{m}x + h$$

Put $\frac{b^2}{4m^2} + 2mh = c$; then we get $h = \frac{4cm^2 - b^2}{8m^3}$, and thence

$$x = \frac{m(h^2 - e)}{md - bh} = \frac{(4m^2c - b^2)^2 - 64m^6e}{8m^2\{4m^2(2m^2d - bc) + b^3\}} \ldots\ldots\ldots\ldots (1).$$

Case II. If e is a square as n^2, then let

$$ax^4 + bx^3 + cx^2 + dx + n^2 = (hx^2 + \frac{dx}{2n} + n)^2$$

$$= h^2x^4 + \frac{dhx^3}{n} + \left(\frac{d^2}{4n^2} + 2nh\right)x^2 + dx + n^2.$$

Put $\frac{d^2}{4n^2} + 2nh = c$, then we get $h = \frac{4cn^2 - d^2}{8n^3}$, and

$$x = \frac{nb - dh}{n(h^2 - a)} = \frac{8n^2\{4n^2(2n^2b - cd) + d^3\}}{(4n^2c - d^2)^2 - 64n^6a} \ldots\ldots\ldots\ldots (2).$$

CASE III. If a is a square, m^2, and e is also a square, n^2; then let

$$m^2x^4+bx^3+cx^2+dx+n^2=(mx^2+hx+n)^2$$
$$=m^2x^4+2mhx^3+(h^2+2mn)x^2+2nhx+n^2.$$

$$\left.\begin{array}{l}\text{And if } 2mh=b;\text{ then } h=\dfrac{b}{2m},\text{ and } x=\dfrac{4m(md-nb)}{b^2+4m^2(2mn-c)},\\[2ex] \quad\text{or if } 2nh=d;\text{ then } h=\dfrac{d}{2n},\text{ and } x=\dfrac{d^2+4n^2(2mn-c)}{4n(nb-md)}\end{array}\right\}\ldots\ldots(3).$$

Cor. If $a=0$, and e is a square, as n^2, then, by the second formula in *Case* III., we have $m=0$, and $x=\dfrac{d^2-4n^2c}{4n^2b}$, which is such a value of x as will render rational the expression

$$\sqrt{(bx^3+cx^2+dx+n^2)}.$$

EXAMPLES.

1. What values of x will render rational the following expressions?

(1) $\sqrt{(3x^2+5x+9)}$. (3) $\sqrt{(2x^2-9x+7)}$.

(2) $\sqrt{(3x^2+8x+5)}$. (4) $\sqrt{(18x^2+33x+13)}$.

(1). By *Prop.* I., Case IV., let $\sqrt{(3x^2+5x+9)}=hx-3$; then squaring, *etc.*, we get

$$x=\frac{6h+5}{h^2-3},\text{ where } h \text{ is arbitrary.}$$

If $h=2$, then $x=17$, and $\sqrt{(3x^2+5x+9)}=31$.

(2). Since $3x^2+8x+5=(3x+5)(x+1)$; therefore by Case V., *Prop.* I., let

$$(3x+5)(x+1)=h^2(x+1)^2,\quad\therefore\quad 3x+5=h^2(x+1);$$

$$\therefore\quad x=\frac{5-h^2}{h^2-3},\text{ where } h \text{ is arbitrary.}$$

If $h=1\frac{4}{5}$ or $\frac{9}{5}$, then $x=\frac{22}{3}=7\frac{1}{3}$, and

$$\sqrt{(3x^2+8x+5)}=15.$$

(3). Since $2x^2 - 9x + 7 = (2x - 7)(x - 1)$; therefore let

$$(2x - 7)(x - 1) = h^2(x - 1)^2, \quad \therefore \quad 2x - 7 = h^2(x - 1);$$

$$\therefore \quad x = \frac{h^2 - 7}{h^2 - 2}, \text{ where } h \text{ is arbitrary.}$$

Let $h = 1$, then $x = 6$, and $\sqrt{(2x^2 - 9x + 7)} = 5$.

(4). Since $18x^2 + 33x + 13 = (4x + 3)^2 + (2x + 1)(x + 4)$; then by Case VI., *Prop.* 1, let

$$(4x+3)^2 + (2x+1)(x+4) = \{(4x+3) + h(2x + 1)\}^2$$
$$= (4x+3)^2 + 2h(4x+3)(2x+1) + h^2(2x+1)^2;$$

$$\therefore x + 4 = 2h(4x + 3) + h^2(2x + 1), \text{ and hence}$$

$$x = -\frac{h^2 + 6h - 4}{2h^2 + 8h - 1}, \text{ where } h \text{ is arbitrary.}$$

Let $h = \frac{1}{3}$, then $x = 1$, and $\sqrt{(18x^2 + 33x + 13)} = 8$.

Supposing we had found by trials that the value $x = 1$, renders $18x^2 + 33x + 13$ an exact square; then by substituting $y + 1$ for x, the expression becomes $18y^2 + 69y + 64$, and by Case IV., *Prop.* 1, let

$$18y^2 + 69y + 64 = (hy - 8)^2 = h^2y^2 - 16hy + 64;$$

$$\therefore y = \frac{69 + 16h}{h^2 - 18}, \text{ where } h \text{ is arbitrary.}$$

Let $h = 3$, then $y = -13$, $x = -12$, and $\sqrt{(18x^2 + 33x + 13)} = 47$.

2. **To divide a given square number, 100, into two such parts, that each of them may be a square number.**

Let a^2 be the given square number, and x^2 one of the parts; then $a^2 - x^2$ is the other part, and since $a^2 - x^2$ is to be a square number, let us assume

$$a^2 - x^2 = (rx - a)^2 = r^2x^2 - 2arx + a^2;$$

whence cancelling, and dividing by x, we get

$$x = \frac{2ar}{r^2 + 1}, \text{ and } rx - a = \frac{2ar^2}{r^2 + 1} - a = a\frac{r^2 - 1}{r^2 + 1};$$

therefore $a^2\left(\frac{2r}{r^2+1}\right)^2$ and $a^2\left(\frac{r^2-1}{r^2+1}\right)^2$ are the parts required, where a and r may be any whole numbers at pleasure.* This is Case IV., *Prop.* 1.

If $a^2 = 100$, and $r = 2$, then the parts are

$$a^2\left(\frac{2r}{r^2+1}\right)^2 = 100 \times \frac{16}{25} = 64, \text{ and } a^2\left(\frac{r^2-1}{r^2+1}\right)^2 = 100 \times \frac{9}{25} = 36.$$

3. To divide a given number, 13, consisting of two known square numbers, 9 and 4, into two other square numbers.

Let $rx - 3 =$ the root of the one square, and $sx - 2 =$ the root of the other; then by the question

$$(rx-3)^2 + (sx-2)^2 = 13.$$

Squaring and cancelling, we obtain $r^2x - 6r + s^2x - 4s = 0$; hence

$$x = \frac{6r+4s}{r^2+s^2},$$

$$rx - 3 = \frac{6r^2+4rs}{r^2+s^2} - 3 = \frac{3r^2+4rs-3s^2}{r^2+s^2} = \text{one root},$$

$$sx - 2 = \frac{6rs+4s^2}{r^2+s^2} - 2 = \frac{2s^2+6rs-2r^2}{r^2+s^2} = \text{the other root}.$$

If $r = 2$, and $s = 1$; then $x = \frac{16}{5}$, $rx - 3 = \frac{17}{5}$, $sx - 2 = \frac{6}{5}$, and

$$13 = 3^2 + 2^2 = \left(\tfrac{17}{5}\right)^2 + \left(\tfrac{6}{5}\right)^2.$$

A variety of answers may be obtained by giving different values to

---

* To this we may add the following useful property.

If s and r be any two unequal numbers, of which s is the greater, it can then be readily shewn, from the nature of the problem, that $2rs$, s^2-r^2, and $(s^2+r^2)^2$ will be the perpendicular, base, and hypothenuse of a right-angled triangle.

From which expressions two square numbers may be found, whose sum or difference shall be square numbers; for $(2rs)^2 + (s^2-r^2)^2 = (s^2+r^2)^2$, and $(s^2+r^2)^2-(2rs)^2 = (s^2-r^2)^2$, or $(s^2+r^2)^2 - (s^2-r^2)^2 = (2rs)^2$; where s and r may be any numbers whatever.

r and s, and if $a^2 + b^2$ be put equal to the number to be divided, the general solution may be obtained in a similar manner.*

4. To find two square numbers which shall have a given difference.

Let d be the given difference, and $(x + r)^2$, and x^2 the required squares;

$$\therefore (x + r)^2 - x^2 \text{ or } 2\,rx + r^2 = d;\ \text{whence}$$

$$x = \frac{d - r^2}{2r},\ \text{and}\ x + r = \frac{d + r^2}{2r}.$$

If $d = 11$, and $r = 1$; then $x = 5$, and $x + r = 6$; or $6^2 - 5^2 = 11$;

if $d = 60 = 30 \times 2$, and $r = 2$, then $x = 14$, and $x + r = 16$, or $16^2 - 14^2 = 60$.

5. To find two numbers such, that their sum and difference shall be both square numbers.

Let $x^2 + 1$ and $2x$ denote the two numbers; then

$$x^2 + 1 + 2x = x^2 + 2x + 1 = (x + 1)^2, \text{ a square.}$$

$$x^2 + 1 - 2x = x^2 - 2x + 1 = (x - 1)^2, \text{ a square.}$$

If $x = 2$, then $x^2 + 1 = 5$, $2x = 4$, and $5 \pm 4 = 9$ or 1,

$x = 3$, then $x^2 + 1 = 10$, $2x = 6$, and $10 \pm 6 = 16$ or 4,

$x = 4$, then $x^2 + 1 = 17$, $2x = 8$, and $17 \pm 8 = 25$ or 9.

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* This problem is considered by DIOPHANTUS as a very important one, being made the foundation of many of his other problems. It may be observed, that in the solution of it, given above, the values of r and s may be taken at pleasure, provided the ratio of them be not the same as that of 3 (a) to 2 (b); the reason of which restriction is, that if r and s were so taken, the sides of the squares sought would come out the same as the sides of the known squares which compose the given number, and therefore the operation would be useless.

The excellent JOHN KERSEY, after amplifying and illustrating this problem in a variety of ways, concludes his chapter thus:—" For a demonstration of the reverse of this *rare* speculation, see ANDERSONUS, Theorem 2, of VIETA'S *mysterious Doctrine of Angular Sections;* and likewise HERIGONIUS, at the latter end of the first *tome* of his *Cursus Mathematicus.*"

6. To find two numbers such that if each be added to the square of the other, the sums shall be square numbers.

Let x, and $2x + 1$ be the two numbers; then

$$x^2 + 2x + 1 = (x + 1)^2 = \text{a square number,}$$

and $(2x + 1)^2 + x = 4x^2 + 5x + 1$, must be a square number.

This may be made a square either by *Case* III. or *Case* IV. of Prop. I., by substitution; but it will be more advantageous to go through the process, as follows:

Let $4x^2 + 5x + 1 = (2x - h)^2 = 4x^2 - 4hx + h^2$; hence

$$(4h + 5)\, x = h^2 - 1, \text{ or } x = \frac{h^2 - 1}{4h + 5};$$

$$\text{and } 2x + 1 = \frac{2\,(h^2 - 1)}{4h + 5} + 1 = \frac{2h^2 + 4h + 3}{4h + 5}.$$

If $h = 4$, then $x = \frac{15}{21} = \frac{5}{7}$, $2x + 1 = \frac{17}{7}$; and

$$(\tfrac{5}{7})^2 + \tfrac{17}{7} = \tfrac{25}{49} + \tfrac{17}{7} = \tfrac{144}{49} = (\tfrac{12}{7})^2,$$

$$(\tfrac{17}{7})^2 + \tfrac{5}{7} = \tfrac{289}{49} + \tfrac{5}{7} = \tfrac{324}{49} = (\tfrac{18}{7})^2.$$

7. To find three numbers such, that not only the sum of all three of them, but also the sum of every two, shall be a square number.

Let $4x$, $x^2 - 4x$, and $2x + 1$ be the three numbers; then since

$$4x + (x^2 - 4x) = x^2,$$

$$(x^2 - 4x) + (2x + 1) = x^2 - 2x + 1 = (x - 1)^2,$$

$$4x + (x^2 - 4x) + (2x + 1) = x^2 + 2x + 1 = (x + 1)^2,$$

we have only to make $4x + (2x + 1) = 6x + 1$, a square.

Let $6x + 1 = a^2$, then $x = \frac{1}{6}(a^2 - 1)$, and the three numbers are

$$\tfrac{2}{3}(a^2 - 1);\ \tfrac{1}{36}(a^2 - 1)^2 - \tfrac{2}{3}(a^2 - 1), \text{ and } \tfrac{1}{3}(a^2 + 2).$$

If $a = 7$, then the numbers are 32, 32, 17;

if $a = 11$, then the numbers are 80, 320, 41.

8. To find three square numbers, such that the sum of every two of them shall be a square number.

Let x^2, y^2, z^2 be the square numbers required; then x^2+y^2, x^2+z^2, and y^2+z^2, are to be squares, or $\frac{x^2}{z^2}+\frac{y^2}{z^2}$, $\frac{x^2}{z^2}+1$ and $\frac{y^2}{z^2}+1$ are to be squares.

Let $\frac{x}{z}=\frac{(3p)^2-1}{6p}$, and $\frac{y}{z}=\frac{p^2-1}{2p}$; then we have

$$\frac{x^2}{z^2}+1=\frac{81p^4+18p^2+1}{36p^2}, \text{ and } \frac{y^2}{z^2}+1=\frac{p^4+2p^2+1}{4p^2},$$

which are both squares, and it only remains to make

$$\frac{x^2}{z^2}+\frac{y^2}{z^2}=\frac{81p^4-18p^2+1}{36p^2}+\frac{p^4-2p^2+1}{4p^2}=\frac{90p^4-36p^2+10}{36p^2}, \text{ a square.}$$

Hence $90p^4-36p^2+10$ must be a square, which it is when $p=1$; let then $p=q+1$, and $90p^4-36p^2+10$ becomes $90q^4+360q^3+504q^2+288q+64$, which must be a square. Let its root be denoted by $tq^2+18q+8$, then we have

$$90q^4+360q^3+504q^2+288q+64=(tq^2+18q+8)^2$$
$$=t^2q^4+36tq^3+(324+16t)q^2+288q+64;$$

and if we make $324+16t=504$, the preceding equation reduces to

$$90q+360=t^2q+36t; \text{ hence we have } t=\frac{180}{16}=\frac{45}{4}, \text{ and}$$

$$q=\frac{360-36t}{t^2-90}=-\frac{16}{13}, \therefore p=q+1=-\frac{3}{13},$$

hence $\frac{x}{z}=\frac{9p^2-1}{6p}=\frac{44}{117}$, and $\frac{y}{z}=\frac{p^2-1}{2p}=\frac{80}{39}=\frac{240}{117}$.

If $z=117$, then $x=44$ and $y=240$, and the required squares are 44^2, 240^2, 117^2 or 1936, 57600 and 13689.

EXAMPLES FOR PRACTICE.

1. It is required to find a number x such, that $x + 1$ and $x - 1$ shall be both square numbers.* Ans. $x = \frac{5}{4}$.

2. It is required to find a number x such, that $x + 128$ and $x + 192$ shall be both squares. Ans. $x = 97$.

3. It is required to find a number such, that $x^2 + x$ and $x^2 - x$ shall be both squares. Ans. $\frac{25}{24}$.

4. It is required to find two numbers such, that if each of them be added to their product, the sums shall be both squares. Ans. $\frac{2}{3}$ and $\frac{5}{3}$.

5. It is required to find three square numbers in arithmetical progression. Ans. 1, 25, and 49

6. To find three whole numbers in arithmetical progression such, that the sum of every two of them shall be a square number.
Ans. 482, 3362, and 6242.

7. To find three numbers such, that if to the square of each the sum of the other two be added, the three sums shall be all squares.
Ans. 1, $\frac{8}{3}$, and $\frac{16}{3}$.

8. To find two numbers in proportion as 8 is to 15, and such that the sum of their squares shall be a square number.
Ans. Any two numbers having the ratio of 8 to 15.

9. To find two numbers such, that if the square of each be added to their product, the sums shall be both squares. Ans. 9 and 16.

10. To find two whole numbers such, that the sum or difference of their squares, when diminished by unity, shall be a square.
Ans. 8 and 9.

11. It is required to resolve 4225, which is the square of 65, into as many other integral squares as the question admits of.
Ans. $16^2 + 63^2$, $56^2 + 33^2$, $60^2 + 25^2$, and $52^2 + 39^2$.

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* The answers to many of the questions here given cannot be found in whole numbers.

12. To find three numbers in geometrical proportion, such that each of them, when increased by a given number (19), shall be square numbers. Ans. 6, 30, and 150.

13. To find two numbers such, that if their product be added to the sum of their squares, the result shall be a square number.
Ans. 5 and 3, 8 and 7, 16 and 5, &c.

14. To find three whole numbers such, that if to the square of each the product of the other two be added, the three sums shall be all squares.
Ans. 9, 73, and 328.

15. It is required to find three square numbers such, that their sum, when added to each of their three roots, shall be all square numbers.
Ans. $\frac{4418}{62920}$, $\frac{13254}{62920}$, and $\frac{18881}{62920}$ = roots required.

16. To find three numbers in geometrical progression such, that if the mean be added to each of the extremes, the sums, in both cases, shall be squares.
Ans. 5, 20, and 80.

17. To find two numbers such, that not only each of them, but also their sum and their difference when increased by unity, shall be all square numbers. Ans. 1368 and 840.

18. To find three numbers such, that whether their sum be added to, or subtracted from, the square of each of them, the numbers thence arising shall be all squares. Ans. $\frac{408}{96}$, $\frac{618}{96}$, and $\frac{781}{96}$.

19. It is required to find three square numbers such, that the sum of their squares shall also be a square number. Ans. 9, 16, and $\frac{144}{25}$.

20. It is required to find three square numbers such, that the difference of every two of them shall be a square number.
Ans. 485809, 34225, and 23409.

21. It is required to divide any given cube number (8) into three other cube numbers. Ans. 1, $\frac{64}{27}$, and $\frac{125}{27}$.

22. To find three square numbers such, that the difference between every two of them and the third shall be a square number.
Ans. 149^2, 241^2, and 269^2.

23. To find three cube numbers such, that if from each of them a given number (1) be subtracted, the sum of the remainders shall be a square number. Ans. $\frac{4913}{3375}$, $\frac{21952}{3375}$, and 8.

EXPONENTIAL EQUATIONS.

An *exponential quantity* is one which has an unknown quantity in its index, as

$$a^x, \ a^{\frac{1}{x}}, \ x^x, \ x^{\frac{1}{x}}, (ax)^x, \textit{ etc.}$$

An *exponential equation* is one which contains any of these exponential quantites in one or more of its terms, as

$$a^x = b, \ x^x = a, \ x^{\frac{1}{x}} = a, \ x^x - x = (2x)^{\frac{1}{x}}, \textit{ etc.}$$

The logarithms of the members of the first of these equations give $x \log a = \log b$;

$$\therefore \ x = \frac{\log b}{\log a}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The second equation, in like manner, becomes

$$x \log x = \log a \quad . \quad . \quad . \quad . \quad (2),$$

and the value of x may be obtained by the aid of a table of logarithms, and the rule for approximating to the roots of equations by *double position*, p. 136.

Taking the logarithms of both members of (2), we get $\log x + \log(\log x) = \log(\log a)$, and if $\log x = z$, and $\log(\log a) = n$; then

$$z + \log z = n \quad . \quad . \quad . \quad . \quad (3);$$

whence the value of z may be obtained by means of a table of logarithms, in the following manner :—

Find that number which added to its logarithm gives a sum as nearly equal to the number n as can be found, and this number will be the value of z or $\log x$; then the number corresponding to the logarithm thus obtained will be the value of x.

EXAMPLES.

1. Given $x^x = 100$, to find the value of x.

Here we have the logarithmic equation

$$x \log x = \log 100 = 2,$$

and since $3{\cdot}5 \times \log 3{\cdot}5 = 3{\cdot}5 \times {\cdot}5440680 = 1{\cdot}904238,$
$3{\cdot}6 \times \log 3{\cdot}6 = 3{\cdot}6 \times {\cdot}5563025 = 2{\cdot}002689$;

hence the value of x is greater than 3·5, but less than 3·6; therefore by the rule, p. 136, we have

$$\begin{array}{ccccc} 2{\cdot}002689 & & 3{\cdot}6 & & 2{\cdot}002689 \\ 1{\cdot}904238 & & 3{\cdot}5 & & 2{\cdot} \\ \hline {\cdot}098451 & : & {\cdot}1 & :: & {\cdot}002689 : {\cdot}00273 ; \end{array}$$

hence $x = 3{\cdot}6 - {\cdot}00273 = 3{\cdot}59727$, nearly.

Again, since $3{\cdot}59727 \times \log 3{\cdot}59727 = 1{\cdot}9999854$
$3{\cdot}59728 \times \log 3{\cdot}59728 = 1{\cdot}9999952,$

$$\begin{array}{ccccc} 1.9999952 & & 3{\cdot}59728 & & 2 \\ 1{\cdot}9999854 & & 3{\cdot}59727 & & 1{\cdot}9999952 \\ \hline {\cdot}0000098 & : & {\cdot}00001 & :: & {\cdot}0000048 : {\cdot}00000485 ; \end{array}$$

hence $x = 3{\cdot}59728 + {\cdot}00000485 = 3{\cdot}5972845$, very nearly.

2. Find the value of x in the equation $ab^x = c$.

Ans. $x = \dfrac{\log c - \log a}{\log b}$.

3. Find the value of x in the equation $a^{b^x} = c$.

Ans. $x = \dfrac{\log d}{\log b}$, where $d = \dfrac{\log c}{\log a}$.

4. Given $a^x . b^y = c$, and $x : y :: m : n$, to find x and y.

Ans. $x = \dfrac{m \log c}{m \log a + n \log b}$, $y = \dfrac{n \log c}{m \log a + n \log b}$.

5. Given $2^{4x} . 3^{2x-5} = 4^{x-1} . 5^{2x}$, to find the value of x.

Ans. $x = 11{\cdot}26242$.

6. Given $(a^2-b^2)^{2(x-1)} = (a-b)^{2x}$, to find the value of x.

Ans. $x = 1 + \dfrac{\log(a-b)}{\log(a+b)}$

7. Given $2^x 3^y = 2000$ and $3y = 5x$, to find x and y.

Ans. $x = \dfrac{3(3 + \log 2)}{3 \log 2 + 5 \log 3}$, $y = \dfrac{5(3 + \log 2)}{3 \log 2 + 5 \log 3}$.

8. Given $3^{2x} + 3^x = 6$, and $4^{2xy} - 2 \times 4^{xy} = 8$, to find x and y.

Ans. $x = \dfrac{\log 2}{\log 3}$, $y = \dfrac{\log 3}{\log 2}$.

9 Given $x^x = 2000$, to find an approximate value of x.

Ans. $x = 4{\cdot}82782263$.

10. Given $(6x)^x = 96$, to find an approximate value of x.

Ans. $x = 1{\cdot}8826432$.

11. Given $x^x = 123456789$, to find an approximate value of x.

Ans. 8·6400268.

12. Given $x^x - x = (2x - x^x)^{\frac{1}{x}}$, to find an approximate value of x.

Ans. $x = 1{\cdot}747933$.

INEQUATIONS.

When two expressions are connected by the sign of inequality, they are treated in a manner similar to equations and on analogous principles.

If the signs of all the terms be changed, the sign of inequality connecting them must also be changed.

Inequations of the first degree.

1. If the inequality be $p > q - r$, by equal additions $p + r > q - r + r$, or finally $p + r > q$.

2. If the inequality be $x < y + z$, then by equal subtractions $x - y < y + z - y$, or $x - y < z$.

3. Whatever be the values of p and q, $p^2 + q^2 > 2pq$.

For whether p or q be the greater $(p-q)^2$ must necessarily be positive, because its square, or $p^2 - 2pq + q^2$, or $p^2 + q^2 - 2pq$ is positive; whence it follows that $p^2 + q^2 > 2pq$.

This mode of reasoning may be usefully employed in numerous other cases.

4. Let it be proved that $p^3 + q^3 > p^2q + pq^2$.

By (3) $p^2 + q^2 > 2pq \therefore p^3 + pq^2 > 2p^2q$, and $p^2q + q^3 > 2pq^2$; whence $p^3 + q^3 + p^2q + pq^2 > 2p^2q + 2pq^2$, or by equal subtractions, $p^3 + q^3 > p^2q + pq^2$.

5. Of the two expressions $p^3 + 1$ and $p^2 + p$, which is the greater, whatever the value of p may be, provided it is > 1?

If $p^3 + 1 > p(p + 1)$, then $p^3+1 > p^2 + p$, whence $p^2-p+1 > p$, by dividing each term by $p + 1$, or since $p^2 + 1 > 2p$ by (3), then $p^3 + 1 > p^2 + p$.

Inequations of the second degree.

Any inequation of the second degree involving one variable may be reduced to

$$x^2 + px + q > 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$x^2 + px + q < 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By Prop. III. theory of equations, page 201 :—

1. $x^2 + px + q = (x - a_1)(x - a_2)$, whence (1) is satisfied by all values of $x >$ than the greatest of a_1 and a_2, or $<$ than the least.

2. In like manner (2) is satisfied by all values of x between a_1 and a_2.

3. When $x^2 + px + q = (x-a_1)^2$, then (1) is satisfied by all values of x except when $x = a_1$.

4. In this case (2) is impossible.

5. When $x^2 + px + q = (x - a_1)^2 + a_2{}^2$, then (1) is satisfied by all values of x.

6. In this case also (2) is impossible.

SCALES OF NOTATION.

Notation is the method of expressing numbers by means of a series of powers of a given number, called the *radix* or *base* of the *scale* in which the numbers are expressed.

In the common system of numeration the radix is 10, and when the radix is 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, *etc.;* the corresponding scale is named the *binary*, *ternary*, *quaternary*, *quinary*, *senary*, *septenary*, *octenary*, *nonary*, *denary* or decimal scale, *undenary*, *duodenary* or duodecimal scale, and so on.

PROP. I. *If r be any integer number, and a, b, c, d, etc., integers each less than r; then any number whatever may be expressed in the form*

$$N = a + br + cr^2 + dr^3 + \dots\dots + hr^{n-1} + kr^n.$$

For divide N by the highest power of r contained in it, as r^n, and let k be the quotient and N_1 the remainder; then $N = kr^n + N_1$. Again divide N_1 by the highest power of r contained in it, as r^{n-1}, and let the quotient and remainder be h and N_2; then $N_1 = hr^{n-1} + N_2$, and $N = kr^n + hr^{n-1} + N_2$. Continue this operation till the remainder becomes less than r, we shall finally arrive at the form enunciated, in which a, b, c, *etc.*, are severally less than r, because the highest possible power of r has been taken in each divisor.

Thus, if $r = 5$ and $N = 279$, then $5^3 = 125$ is contained *twice* in 279 with a remainder 29. Again $5^2 = 25$ is contained *once* in 29, and the remainder is 4; therefore

$$279 = 4 + 1 \cdot 5^2 + 2 \cdot 5^3.$$

PROP. II. *To express a number in any given scale.*

Let r be the radix of the scale, and N the given number; then will

$$\therefore N = a + br + cr^2 + dr^3 + \dots.$$

Now if we divide N by r, the remainder will give a; and the quotient divided by r will give b, and so on till all the digits are obtained.

Note.—In performing these divisions, it is sometimes useful to increase the quotient by unity, in order that the product of r and the increased quotient may differ as little from N as possible. In cases of this kind the remainder is to be accounted *negative*. See *Prop.* IV. *infra.*

EXAMPLES.

1. Express 5271 in the *octenary*, and 10000 in the *nonary* scale.

$$\begin{array}{r} 8\,)\,\underline{5271} \\ 8\,)\,\underline{658}\ldots 7 \\ 8\,)\,\underline{82}\ldots 2 \\ 8\,)\,\underline{10}\ldots 2 \\ 1\ldots 2 \end{array} \qquad\qquad \begin{array}{r} 9\,)\,\underline{10000} \\ 9\,)\,\underline{1111}\ldots 1 \\ 9\,)\,\underline{123}\ldots 4 \\ 9\,)\,\underline{13}\ldots 6 \\ 1\ldots 4 \end{array}$$

$$\therefore (5271)_{10} = (12227)_8, \qquad \text{and } (10000)_{10} = (14641)_9.$$

2. Transform 12227 from the *octenary* to the *ternary* scale.

Here we may first transform the number 12227 from the octenary to the denary or common scale by reversing the process in the preceding Proposition, or by performing the operations implied in the expression for the number.

$$\text{Thus,}\quad 7 + 2.8 + 2.8^2 + 2.8^3 + 8^4 = 5271.$$

The simplest mode, however, is the reversion of the preceding operation, thus:

$$\begin{array}{r} 12227 \\ \underline{8} \\ 10 \\ \underline{8} \\ 82 \\ \underline{8} \\ 658 \\ \underline{8} \\ 5271 \end{array} \qquad\qquad \text{Then}\quad \begin{array}{r} 3\,)\,\underline{5271} \\ 3\,)\,\underline{1757}\ldots 0 \\ 3\,)\,\underline{585}\ldots 2 \\ 3\,)\,\underline{195}\ldots 0 \\ 3\,)\,\underline{65}\ldots 0 \\ 3\,)\,\underline{21}\ldots 2 \\ 3\,)\,\underline{7}\ldots 0 \\ 2\ldots 1 \end{array}$$

$$\text{Hence } (12227)_8 = (5271)_{10} = (21020020)_3.$$

3. In what scale of notation does 738 express 1848?

Let r = the radix of the scale required; then we have

$$7r^2 + 3r + 8 = 1848, \text{ or } 7r^2 + 3r = 1840,$$

a quadratic, which resolved gives $r = 16$ the radix required.

PROP. III. *Every number whatever is composed of the sum of certain terms of the geometrical series*, 1, 2, 2^2, 2^3, 2^4, *etc.*

For in the *binary* scale of notation we have

$$N = a + b.2 + c.2^2 + d.2^3 + e.2^4 + \textit{etc.},$$

where N is any number; and a, b, c, *etc.* are each less than 2, and must therefore be either 1 or 0; hence no term of the progression is taken more than once, and the truth of the proposition is manifest.

Example. Express 39 by means of the powers of 2.

The number 39 expressed in the binary scale is 100111; hence

$$2^0 + 2^1 + 2^2 + 2^5 = 1 + 2 + 4 + 32 = 39.$$

PROP. IV. *Every number whatever is composed of the sum, or of the sum and difference of certain terms of the geometrical series* 1, 3, 3^2, 3^3, 3^4, *etc.*

For in the *ternary* scale of notation, we have

$$N = a + b.3 + c.3^2 + d.3^3 + \textit{etc.},$$

where N is any number, and a, b, c, d, *etc.*, are each less than 2 (see Note, Prop. II), and must therefore be either + 1, −1, or 0; hence no term of the progression is taken more than once, whether it be positive or negative, and therefore the truth of the proposition is manifest.

Example. A person is desirous of weighing 206 lbs. by means of the series of weights, 1 lb., 3 lbs., 9 lbs., 27 lbs., *etc.*; how is it to be done?

The simplest method of doing this is to express 206 in the ternary scale by means of *negative* remainders when necessary. Thus

$$\begin{array}{r l}
3\,)\,206 & \\ \hline
3\,)\,69 \ldots & -1 \\ \hline
3\,)\,23 \ldots & 0 \\ \hline
3\,)\,8 \ldots & -1 \\ \hline
3\,)\,3 \ldots & -1 \\ \hline
1 \ldots & 0
\end{array}$$

Hence $206 = -1 - 3^2 - 3^3 + 3^5 = 3^5 - (3^3 + 3^2 + 1)$, and therefore to weigh 206 lbs. we must place 3^5 or 243 lbs. in one scale, and $3^3 + 3^2 + 1$ or 37 lbs. in the other scale with the substance weighed.*

EXAMPLES FOR PRACTICE.

(*t denotes* 10 *and e* 11 *in these examples.*)

1. Express 400 in the scales of 3, 4, 5, 6, and 12.
Ans. $(400)_{10} = (112211)_3 = (12100)_4 = (3100)_5 = (1504)_6 = (294)_{12}$.

2. Transform 2*t*7*e*5 from the duodenary to the denary scale.
Ans. 59897.

3. The number 215, expressed in a different scale becomes 425; find the radix of the scale? Ans. 7.

4. In what scale of notation does 400 become 484?
Ans. In the nonary.

5. Which of the weights, 1, 2, 2^2, 2^3, *etc.* pounds must be selected to weigh 97 lbs? Ans. $2^6 + 2^5 + 1$.

6. What terms of the series 1, 3, 3^2, 3^3, *etc.* pounds must be employed to weigh 1319 lbs? Ans. $3^7 + 3^4 + 3^3 - (3^6 + 3^5 + 3 + 1)$.

7. If 4954 expressed in a different scale becomes 20305; what is the radix of the scale? Ans. 7.

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* The method of solution here employed, which is new as far as we are aware, is simpler and more direct than that usually given. See **Barlow's** *Theory of Numbers*, or **Hind's** *Algebra*.

PERMUTATIONS AND COMBINATIONS.

The *permutations* of any number of quantities are the different arrangements which can be made with them, when taken *two* together, *three* together, *etc.*, or *all* together.

The *combinations* of any number of quantities are the different sets or groups that can be made of them, when taken *two* together, *three* together, *etc.*, without regard to the order in which they are arranged.

Thus the permutations of the three quantities a, b, c, when taken *three* together, are *abc*, *acb*, *bac*, *bca*, *cab*, *cba*; and the combinations of the four quantities a, b, c, d, when taken *three* together are *abc*, *abd*, *acd*, and *bcd*.

Prop. I. *The number of permutations of n different quantities, taken r together, is* $n(n-1)(n-2)\ldots.(n-r+1)$.

Let the n different quantities be a, b, c, d, *etc.*, and let them be taken *two* together; then a placed first will obviously give $n-1$ permutations; b placed first will give also $n-1$ permutations, and so on, throughout the n different quantities; therefore the total number of permutations is $n(n-1)$, when taken *two* together.

Let the n different quantities be taken *three* together; then the number of permutations of the $n-1$ different quantities, b, c, d, e, *etc.*, is $(n-1)(n-2)$ by the preceding case, when taken *two* together; hence there are $(n-1)(n-2)$ permutations of a, b, c, d, e, *etc.*, taken *three* together in which a stands first; $(n-1)(n-2)$ when b stands first, and so on throughout the n quantities; therefore the total number of permutations of the n different quantities, taken *three* together, is $n(n-1)(n-2)$.

In the same manner, the number of permutations of n quantities taken *four* together is $n(n-1)(n-2)(n-3)$, and generally when taken r together, the number is $n(n-1)(n-2)\ldots.\{n-(r-1)\}$.

If the quantities are taken *all* together, then $r=n$, and the number of permutations is $n(n-1)(n-2)\ldots.3.2.1$ or $1.2.3.4\ldots.n$.

Prop. II. *The number of permutations of n quantities taken all together, of which p quantities are alike, q other quantities are alike, r other quantities are alike, and so on, is*

$$\frac{1 . 2 . 3 . 4 . \ldots\ldots n}{(1 . 2 . 3 \ldots\ldots p)(1 . 2 . 3 \ldots\ldots q)(1 . 2 . 3 \ldots\ldots r) \text{ etc.}}$$

For if the p quantities were unlike, they would form $1 . 2 . 3 . \ldots p$ permutations, instead of only *one* when they are alike; therefore the whole number of permutations must be diminished $1 . 2 . 3 . \ldots\ldots p$ times when p quantities are alike. For a similar reason, the whole number of permutations must be diminished $1 . 2 . 3 \ldots\ldots q$ times when q other quantities are alike, and so on; hence since $1 . 2 . 3 . \ldots\ldots n$ is the whole number of permutations when all the quantities are different, the truth of the proposition is manifest.

Prop. III. *The number of combinations of n different quantities, taken r together, is*

$$\frac{n(n-1)(n-2) \ldots\ldots \{n-(r-1)\}}{1 . 2 . 3 . \ldots\ldots r}.$$

For since each combination of r quantities will form $1.2.3 \ldots\ldots r$ permutations; therefore if c_r denote the whole number of combinations of n quantities taken r together, we shall have (Prop. I.)

$$1.2.3 \ldots\ldots r \times c_r = n(n-1)(n-2) \ldots\ldots \{n-(r-1)\};$$

$$\therefore c_r = \frac{n(n-1)(n-2) \ldots\ldots \{n-(r-1)\}}{1 . 2 . 3 . 4 \ldots\ldots r}.$$

Cor. The sum of all the combinations that can be made of n quantities, taken *one*, *two*, *three*......n together, is

$$c_1 + c_2 + c_3 + \ldots c_n = n + \frac{n(n-1)}{1 . 2} + \frac{n(n-1)(n-2)}{1.2.3} + \text{etc.}$$

$$= (1+1)^n - 1 = 2^n - 1.$$

EXAMPLES.

1. Find the number of permutations of the letters in the term *Algebra.* Here the letter a occurs twice, and $n = 7$; hence (Prop. II.) we have

$$\frac{1.2.3.4.5.6.7}{1.2} = 2520 = \text{number of permutations.}$$

2. In how many different ways can six persons seat themselves at table?
Ans. 720.

3. How many changes may be rung with 5 bells out of 8, and how many with the whole peal? Ans. 6720 and 40320.

4. The number of permutations of n quantities, *three* together, is to the number of permutations, *five* together, as 1 to 42; find n.
Ans. $n = 10$.

5. How many different sums of money can be formed with a penny, a sixpence, a shilling, a crown, and a sovereign? Ans. 31.

6. The number of combinations of n quantities, *four* together, is to the number, *two* together, as 15 to 2; find n the number of quantities.
Ans. $n = 12$.

7. On how many nights may a *different* guard be posted of 4 men out of a company of 36, and on how many of these will any particular soldier be on guard? Ans. 58905 and 6545.

8. Prove that the number of combinations of n quantities, taken r together, is equal to the number taken $n - r$ together.

CONTINUED FRACTIONS.

A *continued fraction* is a complex fraction, composed of a fraction whose denominator is an integer with a fraction, whose denominator is also an integer together with a fraction, and so on. Thus the form of a continued fraction is

$$p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s+\ldots\ldots\ldots,}}}\qquad\text{or written thus: } p+\frac{1}{q+}\,\frac{1}{r+}\,\frac{1}{s+}\ldots\ldots,$$

and $p, p+\frac{1}{q}, p+\frac{1}{q+}\frac{1}{r}, p+\frac{1}{q+}\frac{1}{r+}\frac{1}{s}$, *etc.*, when reduced to their simplest forms, are termed *converging fractions.* The reduced forms of these successive convergents are easily obtained by the following process :—

Let p or $\frac{p}{1}$ be denoted by $\frac{h}{k}$, then we shall evidently have

$$p+\frac{1}{q}=\frac{h}{k}+\frac{1}{q}=\frac{hq+k}{kq}=\frac{h_1}{k_1},\text{ suppose.}$$

Now, if we write $q+\frac{1}{r}$ for q in the last convergent, then we get

$$p+\frac{1}{q+}\frac{1}{r}=\frac{h\left(q+\frac{1}{r}\right)+k}{k\left(q+\frac{1}{r}\right)}=\frac{(hq+k)r+h}{kq.r+k}$$

$$=\frac{h_1r+h}{k_1r+k}=\frac{h_2}{k_2},\text{suppose.}$$

Writing $r+\frac{1}{s}$ for r in the last convergent, gives

$$p+\frac{1}{q+}\frac{1}{r+}\frac{1}{s}=\frac{h_1\left(r+\frac{1}{s}\right)+h}{k_1\left(r+\frac{1}{s}\right)+k}=\frac{h_2s+h_1}{k_2s+k_1}=\frac{h_3}{k_3},$$

and so on; hence any convergent may be derived very simply from the two preceding convergents by the following

Rule.—Multiply the terms of the nth convergent by the $(n+1)$th term of the series p, q, r, s, t, *etc.*, and add to the respective products the terms of the $(n-1)$th convergent.

These convergents are alternately *less* and *greater* than the quantity from which the continued fraction is deduced. Thus the *first* convergent, p or $\frac{p}{1}$, is evidently *less* than the true value; the *second* convergent is too large, for q is too small, and therefore $\frac{1}{q}$ is too great. In like manner, the *third* convergent is shewn to be too small, and so on. Hence also these convergents are successive approximations to the value of the original fraction, or quantity, increasing in accuracy as they increase in number.

Since, by the preceding rule, we have

$$h_4 = h_3 t + h_2, \ \therefore t = (h_4 - h_2) \div h_3,$$
$$k_4 = k_3 t + k_2, \ \therefore t = (k_4 - k_2) \div k_3.$$

Equating these two values of t, and transposing, we get

$$h_4 k_3 - k_4 h_3 = -(h_3 k_2 - k_3 h_2).$$

Similarly, $h_3k_2 - k_3h_2 = -(h_2k_1 - k_2h_1)$; $h_2k_1 - k_2h_1 = -(h_1k - k_1h)$;

$$h_1k - k_1h = (hq + 1) - hq = 1, \text{ since } k = 1;$$

$$\therefore h_2k_1 - k_2h_1 = -1, \ h_3k_2 - k_3h_2 = +1, \text{ and generally}$$

$$h_n k_{n-1} - k_n h_{n-1} = \pm 1,$$

according as n is *odd* or *even*.

This remarkable property of two consecutive convergents furnishes an elegant mode of solution of an indeterminate equation. See Example 2, *infra*.

EXAMPLES.

1. Express $\frac{1848}{365}$ in a continued fraction, and find the convergent fractions.

$$\tfrac{1848}{365} = 5 + \tfrac{23}{365} = 5 + \frac{1}{\frac{365}{23}}; \ \tfrac{365}{23} = 15 + \tfrac{20}{23} = 15 + \frac{1}{\frac{23}{20}};$$

$$\tfrac{23}{20} = 1 + \tfrac{3}{20} = 1 + \frac{1}{\frac{20}{3}}; \ \tfrac{20}{3} = 6 + \tfrac{2}{3} = 6 + \frac{1}{\frac{3}{2}}$$

$\tfrac{3}{2} = 1 + \tfrac{1}{2}$; hence, by successive substitution, we get

$$\tfrac{1848}{365} = 5 + \frac{1}{\frac{365}{23}} = 5 + \tfrac{1}{15} + \frac{1}{\frac{23}{20}} = 5 + \tfrac{1}{15} + \tfrac{1}{1} + \frac{1}{\frac{20}{3}}$$

$$= 5 + \tfrac{1}{15} + \tfrac{1}{1} + \tfrac{1}{6} + \frac{1}{\frac{3}{2}} = 5 + \tfrac{1}{15} + \tfrac{1}{1} + \tfrac{1}{6} + \tfrac{1}{1} + \tfrac{1}{2}.$$

This process is precisely similar to that of finding the greatest common measure of the terms of a fraction, and the quotients 5, 15, 1, 6, 1, 2, are readily obtained by that operation. To find the successive convergents, write down the quotients in a horizontal line, and proceed by the rule before deduced, thus:

$$\begin{array}{cccccccc} & & 5, & 15, & 1, & 6, & 1, & 2, \text{ quotients,} \\ \frac{0}{1}, & \frac{1}{0}, & \frac{5}{1}, & \frac{76}{15}, & \frac{81}{16}, & \frac{562}{111}, & \frac{643}{127}, & \frac{1848}{365}, \text{ converging fractions;} \end{array}$$

where $\frac{0}{1}$ and $\frac{1}{0}$, the symbols of *nought* and *infinity* are to be rejected, having been introduced for the purpose of shewing that the rule for the formation of the convergents applies to *all* the convergents.

2. Given $19x - 117y = 11$, to find integer values of x and y.

By the process of finding the greatest common measure of 19 and 117, we have the following quotients, and thence the convergents:

```
19)117(6
   114
   ———
     3)19(6
       18
       —
        1)3(3
          3
          —
```

$$\begin{array}{ccccc} & & 6 & 6 & 3, \\ \frac{0}{1}, & \frac{1}{0}, & \frac{6}{1}, & \frac{37}{6}, & \frac{117}{19}. \end{array}$$

Hence taking the last convergent $\frac{37}{6}$ and the fraction $\frac{117}{19}$, we have

$$19.37 - 117.6 = 1, \text{ or } 19(37.11) - 117(6.11) = 11;$$

therefore $x = 37.11 = 407$ and $y = 6.11 = 66$, or more generally

$$x = 117m + 407 \text{ and } y = 19m + 66, \text{ where } m \text{ is arbitrary.}$$

If $m = -3$, then $x = 56$ and $y = 9$, which are the least values of x and y that satisfy the equation $19x - 117y = 11$.

3. Reduce $\sqrt{7}$ to a continued fraction.

$$\text{Here } \sqrt{7} = 2 + \sqrt{7} - 2 = 2 + \frac{(\sqrt{7}-2)(\sqrt{7}+2)}{\sqrt{7}+2}.$$

$$= 2 + \frac{3}{\sqrt{7}+2} = 2 + \cfrac{1}{\cfrac{\sqrt{7}+2.}{3}}$$

Again, $\frac{\sqrt{7}+2}{3} = 1 + \frac{\sqrt{7}-1}{3} = 1 + \frac{2}{\sqrt{7}+1} = 1 + \cfrac{1}{\frac{\sqrt{7}+1}{2}}$,

$$\frac{\sqrt{7}+1}{2} = 1 + \frac{\sqrt{7}-1}{2} = 1 + \frac{3}{\sqrt{7}+1} = 1 + \cfrac{1}{\frac{\sqrt{7}+1}{3}},$$

$$\frac{\sqrt{7}+1}{3} = 1 + \frac{\sqrt{7}-2}{3} = 1 + \frac{1}{\sqrt{7}+2} = 1 + \cfrac{1}{\frac{\sqrt{7}+2}{1}},$$

$$\frac{\sqrt{7}+2}{1} = 4 + \frac{\sqrt{7}-2}{1} = 4 + \frac{3}{\sqrt{7}+2} = 4 + \frac{\sqrt{7}+2}{3}, \textit{ etc.}$$

We have now obtained the same expression as we commenced with, and we may therefore discontinue the operation, the denominators 1, 1, 1, 4 recurring in the same order as before; hence by successive substitutions we get

$$\sqrt{7} = 2 + \tfrac{1}{1} + \tfrac{1}{1} + \tfrac{1}{1} + \tfrac{1}{4} + \tfrac{1}{1} + \tfrac{1}{1} + \tfrac{1}{1} + \tfrac{1}{4} + \cdots\cdots$$

The fractions converging to $\sqrt{7}$ are obtained in the usual way, thus:

$$2 \quad 1 \quad 1 \quad 1 \quad 4, \quad 1 \quad 1 \quad 1 \quad 4,$$

$$\tfrac{0}{1},\ \tfrac{1}{0},\ \tfrac{2}{1},\ \tfrac{3}{1},\ \tfrac{5}{2},\ \tfrac{8}{3},\ \tfrac{37}{14},\ \tfrac{45}{17},\ \tfrac{82}{31},\ \tfrac{127}{48},\ \tfrac{590}{223},\ \textit{etc.},$$

each expressing the square root of 7 nearer than any preceding one, and these convergents may be continued at pleasure.

EXAMPLES FOR PRACTICE.

1. Find the series of fractions converging to the fraction $\frac{187}{39}$ or $\frac{39}{187}$.
Ans. 4, 5, $\frac{19}{4}$, $\frac{24}{5}$, or $\frac{1}{4}$, $\frac{1}{5}$, $\frac{4}{19}$, $\frac{5}{24}$.

2. Express 3·1415926535 in a continued fraction, and find the convergents.
Ans. $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, $\frac{103993}{33102}$, $\frac{104348}{33215}$, $\frac{208341}{66317}$.

3. Express $\sqrt{13}$ in a continued fraction, and find the convergents.
Ans. $\frac{3}{1}$, $\frac{4}{1}$, $\frac{7}{2}$, $\frac{11}{3}$, $\frac{18}{5}$, $\frac{119}{33}$, $\frac{137}{38}$, $\frac{256}{71}$, *etc.*

4. Find the fractions converging to the ratio of 365 days 5 hours 48 min. 49 sec. to 29 days 12 hours 44 min. 3 sec.
Ans. $\frac{12}{1}$, $\frac{25}{2}$, $\frac{37}{3}$, $\frac{62}{5}$, $\frac{99}{8}$, $\frac{1349}{109}$, $\frac{1448}{117}$, *etc.*

PROBABILITIES.

The *probability* of an event is the ratio of the number of chances for its happening to the number of chances both for its happening and failing. Thus if an event happen in a ways, and fail in b ways, all being equally probable, the probability of its happening is

$$\frac{a}{a+b} = \frac{\text{the number of favourable events}}{\text{the whole number of events}},$$

and the probability of its failing is

$$\frac{b}{a+b} = \frac{\text{the number of unfavourable events}}{\text{the whole number of events}}.$$

In the theory of probabilities, *certainty* is denoted by unity, and the infinite variety of fractions between 1 and 0, will aptly represent all probabilities whatever. Thus the ratio 1 to 2 or $\frac{1}{2}$ signifies that an event is as likely to happen as to fail; $\frac{3}{4}$ that it is more likely to happen than not, and $\frac{1}{4}$ that it is less likely to happen than to fail.

PROP. I. *The probability that two independent* events will both happen, is equal to the product of the probabilities of the happening of those events considered separately.*

Let a denote the chances of the happening of an event, and b the whole of the chances for its happening and failing; a' the chances of the happening of another independent event, and b' the chances both for its happening and failing; then the separate probabilities of the happening of these events are $\frac{a}{b}$ and $\frac{a'}{b'}$, and those for their failing are $1 - \frac{a}{b}$ and $1 - \frac{a'}{b'}$ or $\frac{b-a}{b}$ and $\frac{b'-a'}{b'}$. Now each of the b events in the first case may be combined with each of the b' events in the second, and thus form bb' combinations (p. 247); and each of the a favourable events in the first case may be combined with each of the a' favourable events in the second, and thus we have aa' favourable events; therefore the probability that both events will happen is expressed by $\frac{a}{b} \times \frac{a'}{b'}$ or $\frac{aa'}{bb'}$. In like manner the probability that both events will not happen is $\frac{b-a}{b} \times \frac{b'-a'}{b'}$ or $\frac{(b-a)(b'-a')}{bb'}$.

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* Events are said to be independent when the happening of any one of them neither increases nor diminishes the probabilities of the happening of the other events.

Or we might reason thus:—Were it *certain* that the first event would happen, then $\frac{a}{b} = 1$, and the probability of the happening of the second event would be $= \frac{a'}{b'}$; but if a is less than b; then the probability of the happening of the second event must be diminished in the ratio of 1 to $\frac{a}{b}$; hence $1 : \frac{a}{b} :: \frac{a'}{b'} : \frac{aa'}{bb'}$, as before.

Cor. 1. The probability that any number of independent events will all happen or will all fail is equal to the product of all the separate probabilities either for happening or failing.

Cor. 2. The probability that both do not happen, though one may, is $1 - \frac{aa'}{bb'}$ or $\frac{bb' - aa'}{bb'}$; the probability that the first will happen and the second fail is $\frac{a(b'-a')}{bb'}$; the probability that the second will happen and the first fail is $\frac{a'(b-a)}{bb'}$; and the probability that one will happen and the other fail, without specifying which event, is the sum of the probabilities that the first will happen and the second fail, and that the second will happen and the first fail, or

$$\frac{a(b'-a')}{bb'} + \frac{a'(b-a)}{bb'} = \frac{ab' + a'b - 2aa'}{bb'} = \frac{a}{b} + \frac{a'}{b'} - \frac{2aa'}{bb'}.$$

Cor. 3. If a always denote the probability of the happening of an event, and b the probability of its happening and failing; then (*Cor.* 1) the fraction $\left(\frac{a}{b}\right)^n$ expresses the probability of its happening n times successively, and $\left(\frac{b-a}{b}\right)^n$ the probability of its failing n times successively.

Prop. II. *To determine the probability that an event will happen a given number of times exactly, in a given number of trials.*

Let a to b be the ratio of the chance of an event's happening to that of its failing; then if n denote the number of trials, and r the number of times the event is to take place, the probability of the event's happening r times successively and failing the remaining $n-r$ times will be (*Prop.* I. *Cor.* 3)

$$\left(\frac{a}{a+b}\right)^r \times \left(\frac{b}{a+b}\right)^{n-r} = \frac{a^r b^{n-r}}{(a+b)^n}.$$

But as there is the same probability of its happening any other r assigned trials, and failing in the rest, it is evident that this probability ought to be repeated as often as n quantities can be combined, r together, which (*Combinations*, p. 247) is

$$= \frac{n(n-1)(n-2)\ldots\ldots\{n-(r-1)\}}{1.2.3\ldots\ldots r};$$

therefore the probability of an event's happening exactly r times in n trials is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3\ldots\ldots r}.\ \frac{a^r b^{n-r}}{(a+b)^n}.$$

Cor. 1. In a similar manner the probability of the event's failing exactly r times in n trials, may be shewn to be

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3\ldots\ldots r}.\ \frac{a^{n-r} b^r}{(a+b)^n}.$$

Cor. 2. The probability of the event's happening at *least* r times in n trials is

$$\frac{a^n + na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 + \ldots\ldots \frac{n(n-1)\ldots\ldots(r+1)a^r b^{n-r}}{1.2.3\ldots\ldots(n-r)}}{(a+b)^n}.$$

For if it happen every time, or fail only once, twice,......$(n-r)$ times, then it happens at least r times; therefore the probability of its happening at *least* r times in n trials is the sum of the probabilities of its happening every time, of failing only once, twice,...... $(n-r)$ times; that is, the probability required is equal to

$$\frac{a^n}{(a+b)^n} + \frac{na^{n-1}b}{(a+b)^n} + \frac{n(n-1)}{1.2}.\frac{a^{n-2}b^2}{(a+b)^n} + \ldots\ldots$$

$$+ \frac{n(n-1)\ldots\ldots(r+1)}{1.2.3\ldots\ldots(n-r)}\ \frac{a^r b^{n-r}}{(a+b)^n}.$$

EXAMPLES.

1. If 5 white and 4 black balls be thrown promiscuously into a bag, what is the probability that the first ball drawn out will be white and the second black? and what is the probability of drawing first a white ball, and then two black ones?

The probability of drawing a white ball first is evidently $\frac{5}{9}$; then the probability of drawing a black ball is $\frac{4}{8}$ or $\frac{1}{2}$, because there are 8 balls left, of which four are white and four black; hence (*Prop.* I.) the probability required is $\frac{5}{9} \times \frac{1}{2} = \frac{5}{18}$. Again, if these two balls are removed, there are left 4 white and 3 black balls, and the probability of drawing another black ball is $\frac{3}{7}$; hence the probability of drawing first a white ball and then two black ones is $\frac{5}{18} \times \frac{3}{7} = \frac{5}{42}$.

2. In how many trials may a person undertake, for an even bet, to throw an ace with a single die?

Let x be the number of trials, then the probability of failing the first time is $\frac{5}{6}$, the probability of failing the second time is also $\frac{5}{6}$, and so on; hence the probability of failing x times together is (*Prop.* I. *Cor.* 3) $(\frac{5}{6})^x$, which by the question must be equal to $\frac{1}{2}$, the probability of its happening; therefore

$$(\tfrac{5}{6})^x = \tfrac{1}{2}, \text{ or } x(\log 5 - \log 6) = \log 1 - \log 2 = -\log 2\,;$$

$$\therefore\ x = \frac{\log 2}{\log 6 - \log 5} = \frac{\cdot 3010300}{\cdot 0791813} = 37\cdot 765 \text{ or } 38 \text{ trials, nearly.}$$

3. If 6 white and 12 black balls are thrown into a bag, what is the probability of drawing a white ball? and what is the probability of drawing two black balls at once? Ans. $\frac{1}{3}$ and $\frac{22}{51}$.

4. Among 32 counters, 12 are red and 20 white, what is the probability of drawing 5 red ones at once; and also the probability of drawing 7 white ones? Ans. $\frac{99}{25172}$ and $\frac{1615}{70122}$.

5. A purse contains 12 pieces of money, of which 4 are sovereigns; what is the probability that in drawing 7 pieces, *three* (and no more) of them will be sovereigns? Ans. $\frac{35}{99}$.

6. In the last example, what is the probability of drawing at *least* three sovereigns among the seven pieces? Ans. $\frac{14}{33}$.

ON SERIES.

THE DIFFERENTIAL METHOD.

The *differential method* is the method of finding the successive differences of the terms of a series, and thence any term of the series, as well as the sum of the whole series.

If the first term of a series be taken from the second, the second from the third, and so on, the remainders will form a new series, called the *first order of differences.* If the terms of this new series be treated as those of the former, the remainders will form a third series, called the *second order of differences*, and so on. Thus if

1, 4, 9, 16, 25, 36, *etc.* be the given series,
3, 5, 7, 9, 11, *etc.* is the 1st order of differences,
2, 2, 2, 2, *etc.* 2d ...
0, 0, 0, 3d ...

Examples. Find the several orders of differences of the following series:

(1) $1^3, 2^3, 3^3, 4^3, 5^3, 6^3, 7^3$, *etc.*
(2) 1, 3, 6, 10, 15, 21, 28, *etc.*
(3) 1, 6, 20, 50, 105, 196, *etc.*
(4) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$, *etc.*

PROBLEM I.

To find the first term of the nth order of differences.

Let a, b, c, d, e, *etc.* be the series; and $d_1, d_2, d_3, \ldots\ldots d_n$ the first terms of the several orders of differences; then the orders of differences are

1st order, $b - a,\ c - b,\ d - c,\ e - d, \ldots\ldots$
2d order, $c - 2b + a,\ d - 2c + b,\ e - 2d + c, \ldots\ldots$
3d order, $d - 3c + 3b - a,\ e - 3d + 3c - b, \ldots\ldots$
4th order, $e - 4d + 6c - 4b + a, \ldots\ldots$
$\vdots$ $\vdots$

whence, by reversing the first terms of these several orders of differences, it is observed that the first term of the nth order of difference is

$$d_n = \pm \left\{ a - nb + \frac{n(n-1)}{1.2} c - \frac{n(n-1)(n-2)}{1.2.3} d + etc. \right\}, \ldots .$$

the sign + being used when n is even, and the sign — when n is odd.

EXAMPLES.

1. Required the first term of the third order of differences of the series 1, 5, 15, 35, 70, *etc.*

Here $a = 1$, $b = 5$, $c = 15$, $d = 35$, *etc.*, and $n = 3$; whence

$$d_3 = - \left\{ a - nb + \frac{n(n-1)}{1.2} c - \frac{n(n-1)(n-2)}{1.2.3} d \right\}$$

$$= -(1 - 15 + 45 - 35) = 4.$$

2. Find the first term of the fourth order of differences of the series 1, 8, 27, 64, 125, *etc.* Ans. $d_4 = 0$.

3. Find the first term of the eighth order of differences of the series 1, 3, 9, 27, 81, *etc.* Ans. $d_8 = 256$.

4. Find the first term of the fifth order of differences of the series

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{32}, \tfrac{1}{64}, etc.$$

Ans. $d_5 = -\frac{1}{32}$.

PROBLEM II.

To find the nth term of a series a, b, c, d, e, etc.

Let d_1, d_2, d_3, *etc.* denote the first terms of the successive orders of differences; then

$$\begin{array}{lll}
d_1 = b - a, & \therefore \; b = a + d_1 & = a + d_1 \\
d_2 = c - 2b + a, & c = -a + 2b + d_2 & = a + 2d_1 + d_2 \\
d_3 = d - 3c + 3b - a, & d = a - 3b + 3c + d_3 & = a + 3d_1 + 3d_2 + d_3 \\
\vdots \quad \vdots & \quad \vdots & \quad \vdots
\end{array}$$

$$\text{Hence the } n\text{th term} = a + (n-1)d_1 + \frac{(n-1)(n-2)}{1.2}d_2 + \frac{(n-1)(n-2)(n-3)}{1.2.3}d_3 + etc.$$

EXAMPLES.

1. Find the twelfth term of the series 2, 6, 12, 20, 30, *etc.*

2, 6, 12, 20, 30,......given series.
4, 6, 8, 10,......1st order of diff.
2, 2, 2,......2d order ...
0 0.......3d order ...

Hence $n = 12$, $d_1 = 4$, $d_2 = 2$, $d_3 = 0$, and therefore

$$a + \frac{n-1}{1}d_1 + \frac{(n-1)(n-2)}{1.2}d_2 = 2 + 11d_1 + 55d_2$$

$$= 2 + 44 + 110 = 156 = \text{the 12th term.}$$

2. Find the nth term, and thence the twentieth term of the series 1, 3, 6, 10, 15, 21, *etc.*

Ans. $\frac{n(n+1)}{2}$, and 210.

3. Find the fifteenth term of the series 1, 4, 9, 16, 25, *etc.*

Ans. 225.

4. Find the thirtieth term of the series 1, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{15}$, $\frac{1}{21}$, $\frac{1}{28}$, *etc.*

Ans. $\frac{1}{465}$.

PROBLEM III.

To find the sum of n terms of a series a, b, c, d, e, etc.

Let us take the series

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d,\ etc. \ldots\ldots (1);$$

then the first order of differences of series (1) is the series

$$a,\ b,\ c,\ d,\ e,\ f,\ etc. \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now it is obvious that the sum of n terms of series (2) is the $(n+1)$th term of series (1); but by *Problem* II., the $(n+1)$th term of series (1) or the sum, s, of n terms of series (2) is evidently

$$s = na + \frac{n(n-1)}{1.2}d_1 + \frac{n(n-1)(n-2)}{1.2.3}d_2, \text{ etc.}$$

EXAMPLES.

1. Find the sum of n terms of the series 1^2, 2^2, 3^2, 4^2, 5^2, *etc.*

1, 4, 9, 16, 25, 36,......given series,
3, 5, 7, 9, 11,......1st order of diff.
2, 2, 2, 2,......2d order ...
0, 0, 0,

$$\text{Hence } s = na + \frac{n(n-1)}{1.2}d_1 + \frac{n(n-1)(n-2)}{1.2.3}d_2$$

$$= n + \frac{3n(n-1)}{2} + \frac{2n(n-1)(n-2)}{1.2.3}.$$

$$= \frac{6n + 9n^2 - 9n + 2n^3 - 6n^2 + 4n}{6} = \frac{2n^3 + 3n^2 + n}{6}$$

$$= \frac{n(n+1)(2n+1)}{1.2.3}.$$

Find the sum of n terms of each of the following series:

	Answers.
2. 2, 6, 12, 20, 30, *etc.*	$\frac{n(n+1)(n+2)}{3}$.
3. 1, 4, 10, 20, 35, *etc.*	$\frac{n(n+1)(n+2)(n+3)}{1.2.3.4}$.
4. 1, 8, 27, 64, 125, *etc.*	$\frac{n^2(n+1)^2}{4}$.
5. 1^4, 2^4, 3^4, 4^4, 5^4, *etc.*	$\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$.
6. 1^5, 2^5, 3^5, 4^5, 5^5, *etc.*	$\frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}$.

INTERPOLATION OF SERIES.

Interpolation of series is the method of inserting any term which may be wanted to complete a given series, or of finding the value of any intermediate term of a given series of equidistant terms.

The interpolation is effected by means of the formulæ deduced in *Problems* I. and II., viz.:—

$$d_n = \pm \left\{ a - nb + \frac{n(n-1)}{1.2}c - \frac{n(n-1)(n-2)}{1.2.3}d + etc. \right\} \text{.....(1).}$$

*n*th term

$$= a + (n-1)d_1 + \frac{(n-1)(n-2)}{1.2}d_2 + \frac{(n-1)(n-2)(n-3)}{1.2.3}d_3 + etc. \text{......(2).}$$

EXAMPLES.

1. Given the logarithms of 101, 102, 104, and 105, to find the logarithm of 103. Supposing the fourth order of differences to vanish, then since in (1) $d = 0$, and $n = 4$, we have

$$a - 4b + 6c - 4d + e = 0,$$

and c is the required term, $\therefore \quad c = \frac{4(b+d) - (a+e)}{6}$.

$$\begin{array}{rlrr}
\text{Now } a = & \log 101 = & 2.0043214 & \\
4b = 4 & \log 102 = \ldots\ldots & & 8.0344008 \\
4d = 4 & \log 104 = \ldots\ldots & & 8.0681332 \\
e = & \log 105 = & 2.0211893 & \\
\cline{3-4}
 & & 4.0255107 & 16.1025340 \\
 & & & 4.0255107 \\
\cline{4-4}
 & & 6\,) & 12.0770233 \\
\cline{4-4}
\therefore & \log 103 = \ldots\ldots\ldots & & 2.0128372
\end{array}$$

2. Given the logarithmic sines of 1°, 1° 1′, 1° 2′, and 1° 3′, to find the log sine of 1° 1′ 40″.

	1° 1′	1° 1′	1° 2′	1° 3′
Log sines	8·2418553	8·2490332	8·2560943	8·2630424
1st order of differences		71779	70611	69481
2d order ...			− 1168	− 1130
3d order ...				38

Hence since 1° 1′ 40″ lies between the second and third terms, and 40″ = $\frac{2}{3}$, we have $n = 2\frac{2}{3} = \frac{8}{3}$, and by (2) the required term is

$$= a + (n-1)d_1 + \frac{(n-1)(n-2)}{1.2}d_2 + \frac{(n-1)(n-2)(n-3)}{1.2.3}d_3$$

$$= a + \tfrac{5}{3}d_1 + \tfrac{5}{9}d_2 - \tfrac{5}{81}d_3$$

$$= 8{\cdot}2418553 + {\cdot}0119632 - {\cdot}0000649 - {\cdot}0000002$$

$= 8{\cdot}2537534 =$ the log sine of 1° 1′ 40″, as required.

3. Given the natural tangents of 88° 54′, 88° 55′, 88° 56′, 88° 57′, 88° 58′, 88° 59′, 89°, to find the natural tangent of 88° 58′ 18″.
Ans. 55·711145.

4. Given the cube roots of 45, 46, 47, 48, and 49, to find the cube root of 50. Ans. 3·684032.

5. Given the logarithms of 50, 51, 52, 54, 55, and 56, to find the logarithm of 53. Ans. 1·72427586.

SUMMATION OF SERIES.

As the summation of certain classes of series has already been given in a former part of the work, we shall only advert to a few examples of summation of some other series. The sum of n terms will be denoted by s_n and the sum of the series *ad infinitum* by the symbol s_∞.

EXAMPLES.

1. Sum n terms of the series $1 + 3 + 7 + 15 + 31 +$ *etc.*

Here

$$s_n = 1+3+7+15+\ldots = (2-1)+(2^2-1)+(2^3-1)+(2^4-1)+\ldots$$

$$= 2+2^2+2^3+2^4+\ldots+2^n-(1+1+1+1+\ldots \text{ to } n \text{ terms})$$

$$= 2\,.\,\frac{2^n-1}{2-1}-n = 2(2^n-1)-n = 2^{n+1}-n-2.$$

2. Sum n terms of the series $\frac{1}{1}+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+$ *etc.*

Here

$$s_n = \tfrac{1}{1}+\tfrac{3}{2}+\tfrac{7}{4}+\tfrac{15}{8}+ etc. = \frac{2-1}{1}+\frac{2^2-1}{2}+\frac{2^3-1}{2^2}+\frac{2^4-1}{2^3}+ etc.$$

$$= 2+2+2+2+\ldots \text{ to } n \text{ terms } -\left(1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\ldots+\frac{1}{2^{n-1}}\right)$$

$$= 2n-\frac{(\frac{1}{2})^n-1}{\frac{1}{2}-1} = 2n-\frac{2^n-1}{2^{n-1}} = \frac{(n-1)2^n+1}{2^{n-1}}.$$

3. Required the sum of the series $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\ldots$ *ad infinitum.*

$$\begin{aligned}
\text{Here}\quad s_\infty &= \tfrac{1}{2}+\tfrac{2}{4}+\tfrac{3}{8}+\tfrac{4}{16}+\tfrac{5}{32}+\ldots\ldots,\\
\therefore\quad \tfrac{1}{2}s_\infty &= \phantom{\tfrac{1}{2}+{}}\tfrac{1}{4}+\tfrac{2}{8}+\tfrac{3}{16}+\tfrac{4}{32}+\ldots\ldots,\\
\hline
\therefore\quad \tfrac{1}{2}s_\infty &= \tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}+\ldots\ldots\\
&= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1;\ \text{therefore } s_\infty = 2.
\end{aligned}$$

4. Find the sum of the series $\frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81} + \ldots\ldots$ *ad infinitum*

Multiplying by $(1-\frac{1}{3})^3$, by the method of detached coefficients, we have

$$\begin{array}{l} \frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81} + \frac{25}{243} + \ldots\ldots \left(-\frac{1}{3} \right. \\ \quad - \frac{1}{9} - \frac{4}{27} - \frac{9}{81} - \frac{16}{243} - \ldots\ldots \\ \hline \frac{1}{3} + \frac{3}{9} + \frac{5}{27} + \frac{7}{81} + \frac{9}{243} + \ldots\ldots \left(-\frac{1}{3} \right. \\ \quad - \frac{1}{9} - \frac{3}{27} - \frac{5}{81} - \frac{7}{243} - \ldots\ldots \\ \hline \frac{1}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \frac{2}{243} + \ldots\ldots \left(-\frac{1}{3} \right. \\ \quad - \frac{1}{9} - \frac{2}{27} - \frac{2}{81} - \frac{2}{243} - \ldots\ldots \\ \hline \end{array}$$

$$\therefore\ (1 - \tfrac{1}{3})^3 s_\infty = \tfrac{1}{3} + \tfrac{1}{9},$$

$$\text{or}\quad (\tfrac{2}{3})^3 s_\infty = \tfrac{4}{9}, \text{ hence } s_\infty = \tfrac{4}{9} \times \tfrac{27}{8} = \tfrac{3}{2} = 1\tfrac{1}{2}.$$

5. Sum the series $\frac{a}{m} + \frac{a+d}{mr} + \frac{a+2d}{mr^2} + \frac{a+3d}{mr^3} + \ldots\ldots$ *ad infinitum.*

Multiplying by $\left(1 - \frac{1}{r}\right)^2$ we have

$$\begin{array}{l} \frac{a}{m} + \frac{a+d}{mr} + \frac{a+2d}{mr^2} + \frac{a+3d}{mr^3} + \ldots\ldots \left(-\frac{1}{r} \right. \\ \quad - \frac{a}{mr} - \frac{a+d}{mr^2} - \frac{a+2d}{mr^3} - \ldots\ldots \\ \hline \frac{a}{m} + \frac{d}{mr} + \frac{d}{mr^2} + \frac{d}{mr^3} + \ldots\ldots \left(-\frac{1}{r} \right. \\ \quad - \frac{a}{mr} - \frac{d}{mr^2} - \frac{d}{mr^3} - \ldots\ldots \\ \hline \end{array}$$

$$\therefore \left(1 - \frac{1}{r}\right)^2 s_\infty = \frac{a}{m} + \frac{d-a}{mr},$$

$$\text{or} \left(\frac{r-1}{r}\right)^2 s_\infty = \frac{a(r-1)+d}{mr},\ \therefore s_\infty = \frac{r}{m}\left\{\frac{a(r-1)+d}{(r-1)^2}\right\}.$$

6. Sum n terms of the series $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} +$ *etc.*

Since the given series $= \dfrac{2}{1.2} + \dfrac{2}{2.3} + \dfrac{2}{3.4} + \dfrac{2}{4.5} + \dots + \dfrac{2}{n(n+1)}$,

and $\dfrac{1}{n} - \dfrac{1}{n+1} = \dfrac{1}{n(n+1)}$; therefore we have

$$\left.\begin{aligned}\tfrac{1}{2}s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\dots + \frac{1}{n}\\ - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \dots\dots - \frac{1}{n} - \frac{1}{n+1}\end{aligned}\right\}$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1};$$

$$\therefore\ s_n = \frac{2n}{n+1} = \text{the sum of } n \text{ terms.}$$

Also, since $s_n = \dfrac{2n}{n+1} = \dfrac{2}{1+\frac{1}{n}}$, when the series is continued *ad infinitum*, the term $\dfrac{1}{n}$ becomes 0; hence $s_\infty = 2$.

7. Find the sum of n terms of the series

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \frac{1}{4.5.6} + \dots\dots + \frac{1}{n(n+1)(n+2)}.$$

Since $\dfrac{1}{n(n+1)} - \dfrac{1}{(n+1)(n+2)} = \dfrac{2}{n(n+1)(n+2)}$; therefore

$$\left.\begin{aligned}2s_n = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots\dots + \frac{1}{n(n+1)}\\ - \frac{1}{2.3} - \frac{1}{3.4} - \frac{1}{4.5} - \dots\dots - \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}.\end{aligned}\right\}$$

$$= \frac{1}{2} - \frac{1}{(n+1)(n+2)}, \text{ or } s_n = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

When n is indefinitely great; then $\dfrac{1}{2(n+1)(n+2)} = 0$, and the sum of the series continued *ad infinitum* is $s_\infty = \frac{1}{4}$.

8. Find the sum of the series $1 + 3x + 6x^2 + 10x^3 + 15x^4 + \dots\dots$ *ad infinitum*, where x is a proper fraction.

Let $s_\infty = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \dots\dots$,

then multiplying both sides by $(1-x)^3$, we have the subjoined operation.

$$\begin{array}{rrrrrl}
1 & +3 & +6 & +10 & +15 & +\dots\dots(-1 \\
 & -1 & -3 & -6 & -10 & - \\
\hline
1 & +2 & +3 & +4 & +5 & +\dots\dots(-1 \\
 & -1 & -2 & -3 & -4 & - \\
\hline
1 & +1 & +1 & +1 & +1 & +\dots\dots(-1 \\
 & -1 & -1 & -1 & -1 & - \\
\hline
\end{array}$$

$\therefore\ (1-x)^3 s_\infty = 1$,

or $s_\infty = \dfrac{1}{(1-x)^3}$ = the sum of the series *ad infinitum.*

9. Find the sum of n terms of the series $1.2 + 2.3 + 3.4 + 4.5 +$ *etc.*, by the method of Indeterminate Coefficients.

Let $1\,.\,2 + 2\,.\,3 + 3\,.\,4 + \dots + n(n+1) = \mathrm{A}n^3 + \mathrm{B}n^2 + \mathrm{C}n \;.\;.\;.\;(1)$,

where the values of A, B, C are independent of n; hence writing $(n+1)$ for n, we get

$$1\,.\,2 + 2\,.\,3 + 3\,.\,4 + \dots\dots + n(n+1) + (n+1)(n+2)$$
$$= \mathrm{A}(n+1)^3 + \mathrm{B}(n+1)^2 + \mathrm{C}(n+1) \;.\;.\;.\;.(2).$$

Subtracting (1) from (2) gives

$$(n+1)(n+2) = 3\mathrm{A}n^2 + (3\mathrm{A} + 2\mathrm{B})n + \mathrm{A} + \mathrm{B} + \mathrm{C};$$

$$\text{or } n^2 + 3n + 2 = 3\mathrm{A}n^2 + (3\mathrm{A} + 2\mathrm{B})n + \mathrm{A} + \mathrm{B} + \mathrm{C}.$$

Equating the coefficients of the same powers of n, we have

$$3\mathrm{A} = 1,\quad 3\mathrm{A} + 2\mathrm{B} = 3,\quad \mathrm{A} + \mathrm{B} + \mathrm{C} = 2;$$

$$\text{hence } \mathrm{A} = \tfrac{1}{3},\qquad \mathrm{B} = 1,\quad \text{and}\quad \mathrm{C} = \tfrac{2}{3};$$

$\therefore$ the sum of n terms of the proposed series is

$$s_n = \mathrm{A}n^3 + \mathrm{B}n^2 + \mathrm{C}n = \frac{n^2}{3} + n^2 + \frac{2n}{3} = \frac{n^3 + 3n^2 + 2n}{3} = \frac{n(n+1)(n+2)}{3}.$$

EXAMPLES FOR PRACTICE.

Find the sums of the following series :—

1. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots\ldots$

Ans. $s_n=\frac{n}{2n+1}$, $s_\infty=\frac{1}{2}$.

2. $\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}+\frac{1}{12.20}+\ldots\ldots$

Ans. $s_n=\frac{n}{12(n+1)}$, $s_\infty=\frac{1}{12}$.

3. $\frac{1}{2.7}+\frac{1}{7.12}+\frac{1}{12.17}+\frac{1}{17.22}+\ldots\ldots$

Ans. $s_n=\frac{n}{2(5n+2)}$, $s_\infty=\frac{1}{10}$.

4. $\frac{1}{2.6}+\frac{1}{4.8}+\frac{1}{6.10}+\frac{1}{8.12}+\ldots\ldots$

Ans. $s_n=\frac{n(3n+5)}{16(n+1)(n+2)}$, $s_\infty=\frac{3}{16}$.

5. $\frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}+\ldots\ldots$

Ans. $s_n=\frac{3}{2}-\frac{3}{(n+1)(n+2)}$, $s_\infty=\frac{3}{2}$.

6. $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\ldots\ldots$

Ans. $s_n=\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}$, $s_\infty=\frac{1}{18}$.

7. $\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+\dots\dots$

Ans. $s_n=\frac{1}{12}-\frac{(-1)^n}{4(2n+3)}$, $s_\infty=\frac{1}{12}$.

8. $\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\frac{8}{4.5.6}+\dots\dots$

Ans. $s_n=\frac{3}{2}-\frac{n+3}{(n+1)(n+2)}$, $s_\infty=\frac{3}{2}$.

9. $\frac{1}{3.6}-\frac{1}{6.8}+\frac{1}{9.10}-\frac{1}{12.12}+\dots\dots$

Ans. $s_n=\frac{1}{24}-\frac{(-1)^n}{12(n+1)(n+2)}$, $s_\infty=\frac{1}{24}$.

In the next two examples x is a proper fraction.

10. $1+4x+10x^2+20x^3+35x^4+\dots\dots$

Ans. $s_\infty=\frac{1}{(1-x)^4}$.

11. $1+8x+27x^2+64x^3+125x^4+\dots\dots$

Ans. $s_\infty=\frac{1+4x+x^2}{(1-x)^4}$.

12. $\frac{1}{5^2-1}+\frac{1}{6^2-1}+\frac{1}{7^2-1}+\frac{1}{8^2-1}+\dots\dots$

Ans. $s_n=\frac{9}{40}-\frac{2n+9}{2(n+4)(n+5)}$, $s_\infty=\frac{9}{40}$.

13. $\frac{1}{3^2-2^2}+\frac{1}{7^2-2^2}+\frac{1}{11^2-2^2}+\frac{1}{15^2-2^2}+\dots$

Ans. $s_n=\frac{n}{4n+1}$, $s_\infty=\frac{1}{4}$.

14. $1.2^2+2.3^2+3.4^2+4.5^2+\dots\dots$

Ans. $s_n=\frac{n^4}{4}+\frac{7n^3}{6}+\frac{7n^2}{4}+\frac{5n}{6}$.

REVERSION OF SERIES.

The *reversion of series* is the method of finding the value of the unknown quantity in that series, by means of another series containing the powers of the quantity which is equal to the given series.

Let $x = a_1 y + b_1 y^2 + c_1 y^3 + etc.$, then to *revert the series*, is to find the value of y in terms of x. Assume $y = bx + cx^2 + dx^3 + etc.$; then by the Table of Factors given at p. 186, the powers of y are readily found to be as follow:

$$y^2 = x^2\{ b^2 + 2bcx + (2bd + c^2)\, x^2 + (2be + 2cd)\, x^3 + \dots\},$$

$$y^3 = x^3\{ b^3 + 3b^2cx + (3b^2d + 3bc^2)\, x^2 + \dots \},$$

$$y^4 = x^4\{ b^4 + 4b^3cx + (4b^3d + 6b^2c^2)\, x^2 + \dots \},$$

$$y^5 = x^5\{ b^5 + 5b^4cx + \dots\}$$

$$\vdots \qquad \vdots$$

Substituting these values of the powers of y in the given equation, and arranging the result, we get

$$x = \begin{array}{r|l|l|l} a_1 \,.\, bx + a_1 \,.\, c & x + a_1 \,.\, d & x^3 + a_1 \,.\, e & x^4 + \dots \\ b_1 \,.\, b^2 & + b_1 \,.\, 2bc & + b_1\, (2bd + c^2) & \\ & + c_1 \,.\, b^3 & + c_1 \,.\, 3b^2c & \\ & & + d_1 \,.\, b^4 & \end{array}$$

Equating the coefficients* of the same powers of x on both sides, we have

---

* The coefficients of the powers of x exceeding the first power, in the left member of the equation, are severally equal to 0.

$$a_1 b = 1, \qquad \therefore b = \frac{1}{a_1}$$

$$a_1 c + b_1 b^2 = 0, \qquad \therefore c = -\frac{b_1}{a^3_1}$$

$$a_1 d + b_1 . 2bc + c_1 b^3 = 0, \qquad \therefore d = \frac{2b^2_1 - a_1 c_1}{a^5_1}$$

$$a_1 e + b_1(2bd + c^2) + c_1 . 3b^2 c + d_1 b^4 = 0, \therefore e = -\frac{5b_1(b^2_1 - a_1 c_1) + a^2_1 d_1}{a^7_1}.$$

etc. *etc.*

Substituting these values for b, c, d, e, *etc.*, and dropping the subscribed figure both in the proposed equation and in these values, as no longer necessary, we find that if $x = ay + by^2 + cy^3 + etc.$, then

$$y = \frac{1}{a}x - \frac{b}{a^3}x^2 + \frac{2b^2 - ac}{a^5}x^3 - \frac{5b\,(b^2 - ac) + a^2 d}{a^7}x^4 + etc. \ldots (\text{A}).$$

EXAMPLES.

1. Given $x = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5 - etc.$, to find y.

Here $a = 1$, $b = -\frac{1}{2}$, $c = \frac{1}{3}$, $d = -\frac{1}{4}$, *etc.*; therefore by (A)

$$y = x + \frac{x^2}{1 . 2} + \frac{x^3}{1 . 2 . 3} + \frac{x^4}{1 . 2 . 3 . 4} + etc.$$

2. Given $x = y - y^2 + y^3 - y^4 + y^5 - etc.$, to find y.

Ans. $y = x + x^2 + x^3 + x^4 + etc.$

3. Given $x = 2y + 3y^3 + 4y^5 + 5y^7 + etc.$, to find y.

Ans. $y = \frac{x}{2} - \frac{3x^3}{16} + \frac{19x^5}{128} - \frac{19x^7}{128} + etc.$

4. Given $x = y - \frac{1}{3}y^3 + \frac{1}{5}y^5 - \frac{1}{7}y^7 + etc.$, to find y.

Ans. $y = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + etc.$

PROPERTIES OF NUMBERS.

Numbers are either prime or composite. A *prime* number is one which has no measure except itself and unity, and all others are *composite* numbers.

Numbers are *prime to each other* when they have no common measure greater than unity. A *perfect* number is one which is equal to the sum of all its divisors.

Let a, b, c, d, *etc.* represent the digits of a number; where a is the digit in the unit's place, b that in the ten's place, c that in the hundred's place, and so on; then if $r = 10$, the general value of any number may be represented by

$$a + br + cr^2 + dr^3 + er^4 + fr^5 + \ldots\ldots (1).$$

Thus $1848 = 8 + 40 + 800 + 1000 = 8 + 4 \,.\, 10 + 8 \,.\, 10^2 + 1 \,.\, 10^3$.

Prop. I. *If from any number the sum of its digits be subtracted, the remainder is divisible by* 9.

For let $a + br + cr^2 + dr^3 + \ldots \qquad =$ the number,

$\therefore \quad a + b + c + d + \ldots \qquad =$ sum of its digits.

Subtracting, we have $b(r-1) + c(r^2 - 1) + d(r^3 - 1) + \ldots$, an expression which is obviously divisible by $r - 1$ or 9. See Example 19, p. 23.

Thus $37452 - (3+7+4+5+2) = 37452 - 21 = 37431 = 9 \times 4159$.

Prop. II. *If the sum of the digits of any number be divisible by* 9, *then the number itself is divisible by* 9.

For let $N = a + br + cr^2 + dr^3 + \ldots$, be the number; then

$$N = a + b(r-1) + b + c(r^2 - 1) + c + d(r^3 - 1) + d + \ldots$$

$$= a + b + c + d + \ldots + b(r-1) + c(r^2 - 1) + d(r^3 - 1) + \ldots$$

Hence N is divisible by $r - 1$ or 9, if $a + b + c + d + \ldots$ is divisible by 9.

Cor. 1. By means of this property, the common process of *casting out nines* in order to test the correctness of a sum in multiplication, may be proved.

Cor. 2. Hence if the sum of the digits of any number be divisible by 3, the number itself is divisible by 3.

PROP. III. *To find a perfect number.*

Since numbers may be represented by the continued multiplication of certain factors, the general form of any number is $a^m b^n c^p d^q \ldots$, where $a, b, c, d \ldots$ are *prime* numbers, and $m, n, p, q, \ldots$ any whole numbers.

Now if $a^m b$ denote a perfect number; then the divisors of $a^m b$ are $1, a, a^2, a^3, \ldots a^m$, and $b, ab, a^2 b, a^3 b, \ldots a^{m-1} b$; hence

$$a^m b = 1 + a + a^2 + \ldots a^m + b + ab + a^2 b + a^3 b + \ldots a^{m-1} b$$

$$= \frac{a^{m+1}-1}{a-1} + b \cdot \frac{a^m - 1}{a-1} = \frac{a^{m+1} + ba^m - b - 1}{a-1};$$

$$\therefore b = \frac{a^{m+1}-1}{a-1} \div \left(a^m - \frac{a^m - 1}{a-1}\right) = \frac{a^{m+1}-1}{a-1} \div \frac{a^{m+1} - 2a_m + 1}{a-1}$$

$$= \frac{a^{m+1} - 1}{a^{m+1} - 2a^m + 1} = \text{a whole number.}$$

Now if $a^{m+1} - 2a^m + 1 = 1$; then we have $a^{m+1} = 2a^m$, or $a = 2$, and $b = 2^{m+1} - 1$,

where $2^{m+1} - 1$ must be a prime number. Let $m = 1, 2, 3, 4,$ *etc.*; then $2^{m+1} - 1 = 3, 7, 15, 31, 63,$ *etc.*, of which 3, 7, 31, 127, *etc.* are *prime* numbers, and 1, 2, 4, 6, *etc.* are the corresponding values of m. Hence if

$$\left.\begin{array}{l}
m = 1,\ a^m b = 2\ (2^2 - 1) = 2 \times 3 = 6 \\
m = 2,\ a^m b = 2^2 (2^3 - 1) = 4 \times 7 = 28 \\
m = 4,\ a^m b = 2^4 (2^5 - 1) = 16 \times 31 = 496 \\
m = 6,\ a^m b = 2^6 (2^7 - 1) = 64 \times 127 = 8128 \\
\qquad\qquad \textit{etc.} \qquad\qquad \textit{etc.}
\end{array}\right\} \text{perfect numbers.}$$

EXAMPLES.

1. If from any number the sum of the digits in the *odd* places be *subtracted,* and to it the sum of the digits in the *even* places be *added,* then the result is divisible by 11.

2. If the sum of the digits in the *even* places be equal to the sum of the digits in the *odd* places, then the number is divisible by 11.

3. The product of any three consecutive numbers is divisible by 6.

4. The product of any four consecutive numbers is divisible by 24.

5. The difference between a number, and the number formed by reversing the order of the digits, is divisible by 9.

6. Prove the arithmetical test for multiplication by *casting out nines.*

INTEREST AND ANNUITIES.

The explanation of the terms used in Interest is given in all treatises on Arithmetic, and need not be repeated here.

An *annuity* is interest of money, rent, or pension, payable periodically, as yearly, half-yearly, quarterly, *etc.*

PROP. I. *To find the amount of a sum of money in any number of years, at simple interest.*

Let P denote the *principal* or sum at interest, r the interest of one pound for one year, t the number of years the principal is at interest, and A the amount of principal and interest at the end of t* years; then since

$$\begin{aligned} r &= \text{the interest of one pound for one year,} \\ tr &= \text{the interest of one pound for } t \text{ years,} \\ \text{P}tr &= \text{the interest of P pounds for } t \text{ years,} \\ \therefore \text{A} &= \text{P} + \text{P}tr = \text{P}(1 + tr) \ldots\ldots\ldots\ldots\ldots (1). \end{aligned}$$

From this formula any one of the four quantities A, P, t, r may be found, when the other three are known.

---

* t may signify the number of *terms*, whether yearly, half-yearly, or quarterly, provided the corresponding part of r be taken.

Prob. II. *To find the amount of a sum of money in any number of years, at compound interest.*

The amount of P pounds in *one* year by *Prob.* I. is $P\,(1+r)$, which becomes the principal for the second year; hence the interest for the second year is $Pr\,(1+r)$, and the amount at the end of *two* years is $= P\,(1+r) + Pr\,(1+r) = P\,(1+r)^2$, which converted into principal for the third year, gives for interest $Pr\,(1+r)^2$; consequently the amount in *three* years is $= P\,(1+r)^2 + Pr\,(1+r)^2 = P\,(1+r)^3$, and so on for any number of years; hence at the expiration of t years, we have

$$A = P\,(1+r)^t, \text{ or in logs, } \log A = \log P + t\log(1+r) \ldots\ldots (2).$$

From this formula any one of the quantities A, P, t, r may be found, when the other three are known.

If the interest is paid *half-yearly*, then t must be changed to $2t$ and r to $\frac{1}{2}r$; hence

$$A = P\,(1+\tfrac{1}{2}r)^{2t}, \text{ or } \log A = \log P + 2t\log(1+\tfrac{1}{2}r) \ldots\ldots (2').$$

If the interest is paid *quarterly*, then, in like manner, we have

$$A = P\,(1+\tfrac{1}{4}r)^{4t}, \text{ or } \log A = \log P + 4t\log(1+\tfrac{1}{4}r) \ldots\ldots (2'').$$

Cor. Hence we can find the number of years in which a sum of money at compound interest will amount to *twice*, *thrice*, or m times itself.

Thus if $A = 2P$, $A = 3P$, or $A = mP$; then by (2)

$$2 = (1+r)^t,\ \ 3 = (1+r)^t, \text{ or } m = (1+r)^t;$$

from which t may be found by the aid of logarithms.

Prob. III. *To find the amount of an annuity in any number of years, at compound interest.*

Let P denote the annuity due at the *end* of the first year; then the amount of the first payment for $t-1$ years is $= P\,(1+r)^{t-1}$; the amount of the second payment for $t-2$ years is $= P\,(1+r)^{t-2}$, and so on to the last payment P at the end of the t years; hence the total amount of the annuity P for t years is equal to the sum of all these partial amounts, and therefore, reversing the steps, we have

$$\begin{aligned} A &= P + P\,(1+r) + P\,(1+r)^2 + \ldots\ldots + P\,(1+r)^{t-1} \\ &= P\,\{1 + (1+r) + (1+r)^2 + \ldots\ldots + (1+r)^{t-1}\}, \end{aligned}$$

$$\text{or } A = P\,\frac{(1+r)^t-1}{(1+r)-1} = P\,.\,\frac{(1+r)^t-1}{r}\;\ldots\ldots\ldots\ldots(3).$$

If the annuity is to be received in *half-yearly* instalments, then we have

$$A = \tfrac{1}{2}P\,.\,\frac{(1+\frac{1}{2}r)^{2t}-1}{\frac{1}{2}r} = P\,.\,\frac{(1+\frac{1}{2}r)^{2t}-1}{r}\;\ldots\ldots(3').$$

If *quarterly*, $A = \tfrac{1}{4}P\,.\,\dfrac{(1+\frac{1}{4}r)^{4t}-1}{\frac{1}{4}r} = P\,.\,\dfrac{(1+\frac{1}{4}r)^{4t}-1}{r}\;\ldots\ldots(3'')$.

Cor. If P pounds are placed out annually for t successive years, and the whole allowed to accumulate at compound interest, then we have, in a similar manner,

$$A = P(1+r) + P(1+r)^2 + P(1+r)^3 + \ldots\ldots P(1+r)^t$$

$$= P(1+r)\,.\,\frac{(1+r)^t-1}{}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

PROB. IV. *To find the present value of an annuity to be paid t years, at compound interest.*

Let $x =$ the present value of the annuity P, then the amount of x pounds in t years is $= x(1+r)^t$ by (2), and the amount of the annuity P in the same time is

$$= P\,.\,\frac{(1+r)^t-1}{r} \text{ by (3) ; hence we obtain}$$

$$x(1+r)^t = P\,\frac{(1+r)^t-1}{r}, \text{ or } x = P\,.\,\frac{(1+r)^t-1}{r(1+r)^t}\;\ldots\ldots\ldots(5).$$

If half-yearly or quarterly payments are made, the corresponding modifications of equation (5) are the same as in the last problem.

If it is a perpetual annuity, or *perpetuity;* then t is infinite, and we have by (5)

$$x = \frac{P}{r}\left\{1 - \frac{1}{(1+r)^t}\right\} = \frac{P}{r}\;\ldots\ldots\ldots\ldots\ldots(5').$$

since $(1+r)^t$ is infinite when t is infinite.

The *purchase of estates* is regulated by the formula given in (5′).

Prob. V. *To find the present value of an annuity to commence at the end of p years, and to continue q years.*

The present value, x, of an annuity restricted to the specified conditions is evidently equal to its present value for $p + q$ years diminished by its present value for p years; that is, by (5)

$$x = \text{P}\frac{(1+r)^{p+q}-1}{r(1+r)^{p+q}} - \text{P}\frac{(1+r)^p-1}{r(1+r)^p} = \frac{\text{P}}{r(1+r)^p}\left\{1-\frac{1}{(1+r)^q}\right\} \text{.........} \quad (6).$$

If the annuity is payable *for ever* after p years have expired; then the *reversion of the perpetuity* is found from (6) by making q infinite.

Now when q is infinite, then $(1+r)^q$ is infinite, and (6) reduces to

$$x = \frac{\text{P}}{r(1+r)^p} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6').$$

EXAMPLES.

1. In how many years will L.400 amount to L.569, 6s. 8d. at 4 per cent. compound interest?

Here $r = \frac{4}{100} = \cdot04$, $\text{P} = 400$, $\text{A} = 569\frac{1}{3} = \frac{1708}{3}$; therefore by (2),

$$t \log 1\cdot04 = \log\frac{1708}{3} - \log 400 = \log 1708 - \log 3 - \log 400,$$

$$\therefore\ t = \frac{\log 1708 - \log 3 - \log 400}{\log 1\cdot04}$$

$$= \frac{3\cdot2324879 - \cdot4771213 - 2\cdot6020600}{\cdot0170333} = \frac{\cdot1533066}{\cdot0170333} = 9 \text{ years, nearly.}$$

2. In what time will any sum of money *double* itself, at 4 per cent. compound interest?

By *Prob.* II. *Cor.* we have $2 = (1 + r)_t$ or $\log 2 = t \log (1 + r)$; therefore since $r = {\cdot}04$, we have

$$t = \frac{\log 2}{\log 1{\cdot}04} = \frac{{\cdot}3010300}{{\cdot}0170333} = 17{\cdot}673 \text{ years.}$$

3. If L.20 is placed out annually for 40 successive years, and allowed to accumulate at the rate of 3½ per cent. compound interest, what would be the amount at the end of 40 years?

Here $\text{P} = 20$, $t = 40$, and $r = \dfrac{3{\cdot}5}{100} = {\cdot}035$; therefore by *Prob.* III. *Cor.* eq. (4),

$$\text{A} = \frac{\text{P}(1+r)}{r}\{(1 + r)^t - 1\},$$

$$\text{or } \log \text{A} = \log \text{P} + \log (1 + r) + \log \{(1 + r)^t - 1\} - \log r.$$

$$\begin{array}{llrll}
\text{Now} & 1 + r = & 1{\cdot}035 & \log = & {\cdot}0149403 \\
 & & & & 40 \\
\cline{5-5}
\therefore & (1 + r)^t = & 3{\cdot}95924 & \log = & {\cdot}5976120 \\
\cline{5-5}
\text{Again} & \text{P} = & 20 & \log = & 1{\cdot}3010300 \\
 & 1 + r = & 1{\cdot}035 & \log = & 0{\cdot}0149403 \\
 & (1 + r)^t - 1 = & 2{\cdot}95924 & \log = & 0{\cdot}4711802 \\
\cline{5-5}
 & & & & 1{\cdot}7871505 \\
 & r = & {\cdot}035 & \log = & \bar{2}{\cdot}5440680 \\
\cline{5-5}
 & \text{A} = & 1750{\cdot}18 & \log = & 3{\cdot}2430825
\end{array}$$

Hence A = L.1750, 3s. 7d. = amount required.

4. Find the present value of an annuity of L.500 for 40 years at 2½ per cent. compound interest.

By *Prob.* IV. eq. (5) we have $x = \dfrac{\text{P}}{r(1 + r)^t}\{(1 + r)^t - 1\}$,

$$\therefore \log x = \log \text{P} + \log \{(1 + r)^t - 1\} - \log r - t \log (1 + r).$$

$$\begin{array}{lrrlrr}
\text{Now} & 1+r = & 1{\cdot}025 & \log = & \underline{{\cdot}0107239 \atop 40} & \\
\therefore & (1+r)_t = & \underline{2{\cdot}68507 \atop 1} & \log = & \underline{{\cdot}4289560} & \\
\therefore & (1+r)^t - 1 = & 1{\cdot}68507 & \log = & {\cdot}2266180 & \\
 & p = & 500 & \log = & 2{\cdot}6989700 & \\
 & r = & {\cdot}025 & \log = & \ldots\ldots & \bar{2}{\cdot}3979400 \\
 & (1+r)^t = & (1{\cdot}025)^{40} & \log = & \ldots\ldots & \underline{0{\cdot}4289560} \\
 & & & & 2{\cdot}9255880 & \bar{2}{\cdot}8268960 \\
 & & & & \underline{\bar{2}{\cdot}8268960} & \\
 & x = & 12551{\cdot}4 & \log = & 4{\cdot}0986920 &
\end{array}$$

Hence the present value is = L.12551, 8s., nearly.

5. Find the amount of L.450 at 5 per cent. compound interest, for 3 years, both when the interest is to be paid yearly and half-yearly.

Ans. L.520, 18s. 7½d., and L.521, 17s. 2½d.

6. If a freehold estate be purchased for L.9520, what is the yearly rent, allowing the purchaser 4 per cent. for his money?

Ans. L.360, 16s.

7. What is the present value of an annuity of L.20 for 25 years, to commence at the expiration of 5 years, allowing 4 per cent. compound interest?

Ans. L.256, 16s. 1d.

8. Suppose an annuity of L.175 is to commence 9 years hence, and then continue 11 years; find its present value, allowing 6 per cent. compound interest.

Ans. L.816, 18s. 9d.

9. Required the compound interest and amount of L.160 in 4 years, at 6 per cent. per annum.

Ans. Interest, L.41, 19s. 11d.
Amount, L.201, 19s. 11d.

10. If P pounds at compound interest, rate r, double itself in h years, and treble itself at rate $2r$ in k years; find the relation between h and k.

$$\text{Ans. } \frac{h}{k} = \frac{\log 2}{\log 3} \cdot \frac{\log(1+2r)}{\log(1+r)}.$$

11. The present value of a freehold estate of L.100 per annum, subject to the payment of an annuity at the end of every two years, is L.1000; find the annuity, allowing 5 per cent. compound interest.

Ans. L.102, 10s.

12. A debt of P pounds accumulating at compound interest is discharged in h years by annual payments of $\frac{\text{P}}{m}$ pounds; shew that, if r be the interest of one pound for a year,

$$h = -\frac{\log(1-mr)}{\log(1+r)}.$$

13. A sum of money, P pounds, is left among A, B, C, in such a manner that at the end of a, b, c years, when they respectively come of age, they are to possess equal sums; find the share of each, allowing compound interest at rate r.

$$\text{Ans. A's share} = \frac{\text{P}}{1+(1+r)^{a-b}+(1+r)^{a-c}}.$$

$$\text{B's share} = \frac{\text{P}}{1+(1+r)^{b-a}+(1+r)^{b-c}}.$$

$$\text{C's share} = \frac{\text{P}}{1+(1+r)^{c-a}+(1+r)^{c-b}}.$$

14. The present value of an annuity of L.1 to continue x years is L.10, and to continue $2x$ years is L.16; required the rate of interest.

Ans. 4 per cent.

15. Four persons A, B, C, D contribute equal sums towards the purchase of a freehold estate; find the terms that A, B, C may successively enjoy it, in order that D may be entitled to the absolute reversion.

$$\text{Ans. (A) } \frac{\log 4 - \log 3}{\log(1+r)},\ \text{(B) } \frac{\log 3 - \log 2}{\log(1+r)},\ \text{(C) } \frac{\log 2}{\log(1+r)} \text{ years.}$$

THEORY OF POPULATION.

Let P denote the population of a country at any given period, $\frac{1}{m}$ the fractional part of the population which die in a year, or the ratio of mortality, $\frac{1}{b}$ the ratio of births in a year, and A the amount of the population at the end of n years; then

$\frac{1}{b} - \frac{1}{m} = \frac{m-b}{mb} =$ rate of increase of population in one year,

and $P + \frac{m-b}{mb}P = P\left(1 + \frac{m-b}{mb}\right) =$ population at the end of the first year.

Now since P is changed to $P\left(1 + \frac{m-b}{mb}\right)$, we must write this last value for P in the preceding state of the population at the end of one year, and we get

$P\left(1 + \frac{m-b}{mb}\right)^2 =$ population at the end of two years; hence at the expiration of n years, we must have

$$A = P\left(1 + \frac{m-b}{mb}\right)^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

from which if four of the involved quantities be given, the remaining one may be found.

But since, in any country, under *given* circumstances of births and deaths, the fraction $\frac{m-b}{mb}$ is always a *given* quantity, we may represent it by $\frac{1}{p}$; then (1) becomes

$$A = P\left(1 + \frac{1}{p}\right)^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1')$$

Suppose the population is increased q times; then $\text{A} = q\text{P}$, and we have the relation

$$q = \left(1 + \frac{1}{p}\right)^n \text{..............................} (2).$$

EXAMPLES.

1. The population of Great Britain in 1841 was 27019558; now if the $\frac{1}{40}$th part *die* annually, and the births be to the deaths as 40 to 30, what will be the amount of the population of Great Britain in the year 1900?

Here $\text{P} = 27019558$, $n = 59$, $m = 40$, $b = 30$, and therefore

$$\frac{1}{p} = \frac{1}{b} - \frac{1}{m} = \frac{m-b}{mb} = \frac{1}{120}, \text{ and } 1 + \frac{1}{p} = \frac{121}{120}; \text{ hence } (1')$$

$$\log \text{A} = \log \text{P} + n \log\left(1 + \frac{1}{p}\right)$$

$$= \log 27019558 + 59\,(\log 121 - \log 120)$$

$$= 7{\cdot}4316783 + 59 \times {\cdot}0036042$$

$$= 7{\cdot}6443261 = \log 44088570.$$

Hence $\text{A} = 44088570 =$ population in the year 1900.

2. What must be the annual increase of population in any country, that it may be *doubled* in 100 years?

By formula (2) we have $q = 2$, and $n = 100$; hence

$$\log 2 = 100 \log\left(1 + \frac{1}{p}\right); \text{ therefore}$$

$$\log\left(1 + \frac{1}{p}\right) = \frac{\log 2}{100} = \frac{{\cdot}3010300}{100} = {\cdot}0030103$$

$$= \log 1{\cdot}0069556; \text{ consequently}$$

$$1 + \frac{1}{p} = 1{\cdot}0069556, \text{ or } \frac{1}{p} = {\cdot}0069556 = \frac{1}{144}, \text{ nearly.}$$

3. If the number of persons born in any year be $\frac{1}{45}$ of the whole population at the commencement of that year, and the number of those who die be $\frac{1}{60}$ of it; in how many years will the population be doubled?
Ans. In 125 years.

4. Supposing the annual increase of a province containing 1800000 inhabitants to be $\frac{1}{30}$ of the whole; what will be the population at the end of a century?
Ans. 47787300 of inhabitants.

5. Suppose the population of an empire to be 40 millions, and the annual increase $\frac{1}{195}$th; in how many years will it amount to 50 millions?
Ans. In 43·6 years.

6. What must be the rate of increase that a population of 1106400 persons may become five millions in 100 years?
Ans. $\frac{1}{66}$ annually.

7. The population of a province in 1801 was 500000, and in 1841 it amounted to 720000; now supposing the ratio of mortality to be $\frac{1}{50}$, what was the ratio of births?
Ans. $\frac{1}{34}$, nearly.

8. The population of the British Isles in 1841 was 27019558; in 1831 it was 24410429; required the value of $\frac{1}{p}$,* and thence the population in 1851.

Ans. $\frac{1}{p}=\frac{1}{100}$ very nearly, and in 1851 the population will be 29907555.

9. The population of France in 1841 was 34230178; the value of m was 50, that of b 40; what may be the expected population of France in 1851?
Ans. 35980700.

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* Note.—It would perhaps be convenient to put $\frac{1}{p}=r$, the rate of increase, to render the preceding analogous to the formulæ for Interest and Annuities.

APPLICATION OF ALGEBRA

TO THE

RESOLUTION OF GEOMETRICAL PROBLEMS.

As the characters which are employed in Algebra may represent every class of magnitudes, the relations of lines, surfaces, *etc.*, may be deduced, and properties of them discovered, with more facility by the principles of Algebra, under the guidance of Geometry, than by purely geometrical research.

When a geometrical problem is proposed for solution, a figure is to be drawn that shall represent the conditions of the problem, and this figure is to be regarded as a correct one. Then consider the nature of the problem, and, if necessary, draw such other lines in the figure as may appear to facilitate the solution of the problem under consideration. Denote the known and unknown quantities by the usual algebraic characters, and form as many independent equations as there are unknown quantities to be determined; then find the values of these unknown quantities by the ordinary principles of Algebra.

The following directions may be useful to the student.

1. Having drawn a figure representing the conditions of the problem, draw other lines, if necessary, either parallel or perpendicular to those of the figure, or so as to form *similar triangles*.

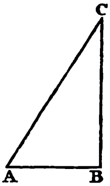
2. When an angle is given, draw a perpendicular opposite to it from the extremity of one of the lines containing the given angle.

3. Denote the known or given quantities by a, b, c, *etc.*, and the unknown or required quantities by z, y, x, *etc.*

4. When two lines or quantities are alike related to other parts of the figure, it is often convenient to substitute for their sum or difference, or rectangle, or for some quantity to which they have both the same relation.

PROBLEM I.

Given one side of a right-angled triangle $= 5$, *and the sum of the hypothenuse and the other side* $= 25$, *to determine the hypothenuse and the remaining side.*



Let ABC represent the required triangle, the angle ABC being a right angle, and let AB $= a$, the sum of AC and BC $= b$, and BC $= x$; then AC $= b - x$, and (Euc. I. 47,)

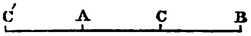
$$\text{AB}^2 + \text{BC}^2 = \text{AC}^2, \text{ or } a^2 + x^2 = (b-x)^2 = b^2 - 2bx + x^2;$$

$$\therefore \quad 2bx = b^2 - a^2, \text{ and } x = \frac{b^2 - a^2}{2b} = \frac{25^2 - 5^2}{50} = 12 = \text{BC}.$$

$$\text{Also, } b - x = b - \frac{b^2 - a^2}{2b} = \frac{b^2 + a^2}{2b} = \frac{25^2 + 5^2}{50} = 13 = \text{AC}.$$

PROBLEM II.

To divide a given straight line into two parts, such that the square of the greater part may be equal to the rectangle contained by the whole line and the less part.



Let the given line AB $= a$, and the greater part AC $= x$, then the less part BC $= a - x$, and by the condition of the problem we have

$$x^2 = a(a-x) = a^2 - ax, \text{ or } x^2 + ax = a^2.$$

The solution of this quadratic gives the two values

$$x = -\frac{a}{2} \pm \frac{a}{2}\sqrt{5} = \frac{a}{2}(-1 \pm \sqrt{5}),$$

the one being positive and the other negative.

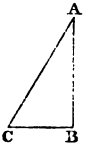
This problem furnishes an instance of the geometrical interpretation of a negative value of an algebraic quantity. For in the preceding solution we supposed AC to extend from A towards B, and hence the positive value of x is AC; but the line corresponding to the negative value of x must lie in a direction opposite to that of AC, and therefore if BA be *produced* to C′, so that AC′ may be equal to $-\frac{1}{2}a(1+\sqrt{5})$, the negative value of x, then AC′ will represent the negative value of x.

The interpretation of a negative result may generally be ascertained by writing $-x$ for x in the original equation, a transformation which simply changes the signs of the roots of the equation, without altering their values. Thus in the quadratic found above, viz., $x^2 + ax = a^2$, substitute $-x$ for x, then we have $x^2 - ax = a^2$, or $x^2 = a^2 + ax = a(a + x)$, which leads to the following problem.

To produce a given straight line, so that the square of the produced part shall be equal to the rectangle contained by the given line and the line which is made up of the whole and the part produced.

PROBLEM III.

The hypothenuse of a right-angled triangle is 13, *and the excess of the perpendicular above the base is* 7 ; *find the sides of the triangle.*



Let the base BC $= x$, then the perpendicular AB $= x + 7$, and therefore, by Euc. I. 47, we have the equation

$$AB^2 + BC^2 = AC^2, \text{ or } (x + 7)^2 + x^2 = 13^2;$$

and expanding and transposing we get

$$2x^2 + 14x = 169 - 49 = 120;$$

$$\therefore \quad x^2 + 7x = 60;$$

which resolved by quadratics gives either $x = 5$, or $x = -12$; hence

$$AB = 12 \text{ and } BC = 5.$$

PROBLEM IV.

The hypothenuse of a right-angled triangle is 13, *and the sum of the base and perpendicular is* 17 ; *find the sides of the triangle.*

Let the base BC $= x$, then AB $= 17 - x$, and therefore (Euc. I. 47)

$$x^2 + (17 - x)^2 = 13^2,$$
$$\text{or } 2x^2 - 34x = 169 - 289 = -120,$$
$$\therefore\ x^2 - 17x = -60,$$

and the resolution of this quadratic gives either $x = 12$, or $x = 5$; hence the sides containing the right angle are 5 and 12.

PROBLEM V.

The hypothenuse of a right-angled triangle is 13, *and the product of the base and perpendicular is* 60; *find the sides of the triangle.*

Let BC $= x$, and AB $= y$; then the equations of condition, by Euc. I. 47, and the hypothesis, are

$$x^2 + y^2 = 169 \ldots\ldots(1) \qquad xy = 60 \ldots\ldots(2)$$

To and from equation (1) add and subtract twice eq. (2); then we get

$$x^2 + 2xy + y^2 = 289 \quad \therefore\ x + y = \pm 17$$
$$x^2 - 2xy + y^2 = 49 \quad \therefore\ x - y = \pm 7.$$

Consequently by adding and subtracting the latter of these equations from the former, we have $x = 12$, and $y = 5$.

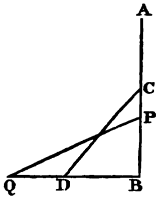
PROBLEM VI.

The top of a maypole, broken by the wind, struck the ground twenty feet from the base, and being fixed, was again broken five feet lower than before, and its top then struck the ground thirty feet from the base; find the height of the maypole.

Let AB be the maypole, BC and CD the standing and broken parts in the first case, and BP, PQ the corresponding parts in the second case. Let

BD $= a$, BQ $= b$, CP $= c$, BC $= x$, AC $=$ CD $= y$;

then BP $= x - c$, and AP $=$ PQ $= y + c$; hence



(Euc. I. 47)...BD^2 + BC^2 = CD^2, or $a^2 + x^2 = y^2 \ldots(1)$

...BQ^2 + BP^2 = PQ^2, or $b^2 + (x - c)^2 = (y + c)^2 \ldots\ldots(2).$

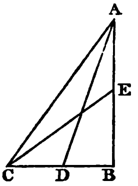
Expanding (2), and subtracting equation (1) from it, we have

$$b^2 - a^2 - 2cx = 2cy, \text{ or } b^2 - a^2 = 2c(x + y);$$

$$\therefore \quad x + y = \frac{b^2 - a^2}{2c} = 50 \text{ feet} = \text{the height AB.}$$

PROBLEM VII.

In a right-angled triangle, the lengths of the lines drawn from the acute angles to the points of bisection of the opposite sides are a and b ; find the sides of the triangle.



Let ABC be the triangle, AD and CE the lines drawn from the acute angles to the points of bisection D and E of the opposite sides. Let AD $= a$, CE $= b$, BC $= 2x$, AB $= 2y$; then

$$\text{(Euc. I. 47)}\ldots \text{AD}^2 = \text{AB}^2 + \text{BD}^2, \text{ or } a^2 = x^2 + 4y^2 \ldots (1)$$

$$\ldots \text{CE}^2 = \text{BC}^2 + \text{BE}^2, \text{ or } b^2 = 4x^2 + y^2 \ldots (2)$$

By adding and subtracting these two equations, we get

$$y^2 + x^2 = \frac{a^2 + b^2}{5} \ldots\ldots (3), \quad y^2 - x^2 = \frac{a^2 - b^2}{3} \ldots\ldots\ldots (4),$$

$$\therefore 4y^2 = \frac{4(4a^2 - b^2)}{15}, \text{ or } 2y = \text{AB} = 2\sqrt{\left(\frac{4a^2 - b^2}{15}\right)},$$

$$4x^2 = \frac{4(4b^2 - a^2)}{15}, \text{ or } 2x = \text{BC} = 2\sqrt{\left(\frac{4b^2 - a^2}{15}\right)}.$$

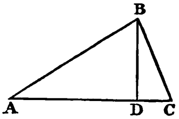
Hence also $\text{AC}^2 = \text{AB}^2 + \text{BC}^2 = \frac{4}{15}(3a^2 + 3b^2) = \frac{4}{5}(a^2 + b^2)$

$$\text{or AC} = 2\sqrt{\left(\frac{a^2 + b^2}{5}\right)} = \text{the hypothenuse.}$$

PROBLEM VIII.

Given the perimeter of a right-angled triangle, and the perpendicular drawn from the right angle to the hypothenuse, to determine the sides of the triangle.

Let BD $= p$, AB + BC + CA $= 2s$, AB $= x$, BC $= y$, and AC $= z$; then



(Euc. VI. 8) .... $xy = pz$ .... (1)
(„ I. 47) ...... $x^2 + y^2 = z^2$ .... (2)
(hypothesis) ......$x+y+z = 2s$ .... (3)

Multiply (1) by 2, and add the product to equation (2); then

$$x^2 + 2xy + y^2 = z^2 + 2pz, \text{ or } (x+y)^2 = z^2 + 2pz.$$

But by (3) we have $x + y = 2s - z$; consequently

$$z^2 + 2pz = (2s-z)^2 = 4s^2 - 4sz + z^2;$$

$$\therefore z = \frac{4s^2}{4s+2p} = \frac{2s^2}{2s+p} = \text{AC, the hypothenuse.}$$

$$\text{Again, by (3), } x + y = 2s - z = 2s - \frac{2s^2}{2s+p} = \frac{2s(s+p)}{2s+p};$$

$$\text{and (1)} \quad xy = pz = \frac{2s^2p}{2s+p};$$

$$\text{whence we obtain } x - y = \frac{2s}{2s+p}\sqrt{s^2-2sp-p^2};$$

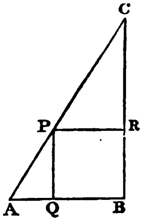
$$\therefore \quad x = \frac{s}{2s+p}\left(s + p + \sqrt{s^2-2sp-p^2}\right) = \text{the side AB,}$$

$$\text{and } y = \frac{s}{2s+p}\left(s + p - \sqrt{s^2-2sp-p^2}\right) = \text{the side BC.}$$

PROBLEM IX.

Given the hypothenuse of a right-angled triangle, a, and the side of a square inscribed in it, so that one of its angles may be coincident with the right angle of the triangle, b; to determine the sides of the triangle.

Let ABC be the right-angled triangle, and PRBQ the square inscribed in it, as specified in the enunciation of the problem. Let



AB $= x$, and BC $= y$; then CR $= y - b$, and hence

(Euc. I. 47) ...... $x^2 + y^2 = a^2$ .... (1)

(„ VI. 4), AB : BC :: PR : RC, or $x : y :: b : y - b$;

$$\therefore b(x+y) = xy \ldots\ldots (2)$$

To eq. (1) add twice eq. (2), and we get at once, after transposition,

$$(x+y)^2 - 2b(x+y) = a^2;$$

from which, by the resolution of a quadratic, we have

$$x + y = b \pm \sqrt{a^2 + b^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3);$$

$$\therefore\ xy = b(x+y) = b^2 \pm b\sqrt{a^2 + b^2} \ldots\ldots (4).$$

From (1) take twice equation (4), and extract the root of the remainder,

$$x - y = \pm \{a^2 - 2b^2 \mp 2b\sqrt{a^2 + b^2}\}^{\frac{1}{2}} \ldots\ldots\ldots (5).$$

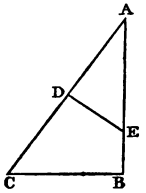
Taking half the sum, and then half the difference of equations (3, 5), we get

$$x = \tfrac{1}{2}(b \pm \sqrt{a^2 + b^2}) \pm \tfrac{1}{2}(a^2 - 2b^2 \mp 2b\sqrt{a^2 + b^2})^{\frac{1}{2}} = \text{AB},$$

$$y = \tfrac{1}{2}(b \pm \sqrt{a^2 + b^2}) \mp \tfrac{1}{2}(a^2 - 2b^2 \mp 2b\sqrt{a^2 + b^2})^{\frac{1}{2}} = \text{BC}.$$

PROBLEM X.

The hypothenuse of a right-angled triangle is $= 2a$, *and a perpendicular from its middle point cuts from the perpendicular a segment adjacent to the right angle* $= b$; *find the sides of the triangle.*



Let DE be the perpendicular from the middle of the hypothenuse, and BE the segment cut from the perpendicular AB. Then AD $=$ DC $= a$, BE $= b$, and let AE $= x$; then we have AB $= x + b$, and the triangles ABC, ADE are equiangular; therefore

$$\text{AB} : \text{AC} :: \text{AD} : \text{AE}, \text{ or } x + b : 2a :: a : x;$$

$$\therefore\ x^2 + bx = 2a^2,$$

and resolving the quadratic we obtain either

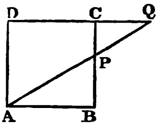
$$x = -\frac{b}{2} + \frac{\sqrt{b^2 + 8a^2}}{2}, \text{ or } x = -\frac{b}{2} - \frac{\sqrt{b^2 + 8a^2}}{2};$$

hence the sides of the triangle are

$$\text{AB} = \frac{b}{2} + \frac{\sqrt{b^2 + 8a^2}}{2}; \text{ and BC} = \tfrac{1}{2}\sqrt{\{8a^2 - 2b^2 - 2b\sqrt{8a^2 + b^2}\}}.$$

PROBLEM XI.

From one of the angular points of a given square, to draw a straight line such, that the part of the line intercepted between the two sides of the square containing the opposite angle may be equal to a given line.



Let ABCD be the given square, and APQ the required line of which the intercepted part is PQ. Let

AB $= a$, PQ $= b$, and BP $= x$; then CP $= a - x$, and

(Euc. I. 47)...AB^2 + BP^2 = AP^2, or $\sqrt{a^2 + x^2} =$ AP.

(Euc. VI. 4)...BP : PA :: CP : PQ, or $x : \sqrt{a^2 + x^2} :: a - x : b$;

$$\therefore\quad b^2x^2 = (a - x)^2 (a^2 + x^2) = a^4 - 2a^3x + 2a^2x^2 - 2ax^3 + x^4,$$

hence, $$x^4 - 2ax^3 + (2a^2 - b^2)x^2 - 2a^3x + a^4 = 0,$$

$$\text{or } (x^4 + a^4) - 2ax(x^2 + a^2) + (2a^2 - b^2)x^2 = 0.$$

But since $x^4 + a^4 = (x^2 + a^2)^2 - 2a^2x^2$; therefore the last equation becomes

$$(x^2 + a^2)^2 - 2ax(x^2 + a^2) = b^2x^2;$$

which being resolved as a quadratic gives

$$x^2 - (a \pm \sqrt{a^2 + b^2})x = - a^2,$$

and this resolved again gives

$$x = \frac{a \pm \sqrt{a^2 + b^2}}{2} \pm \frac{\sqrt{b^2 - 2a^2 \pm 2a\sqrt{a^2 + b^2}}}{2}.$$

The equation deduced above may be resolved in a different manner. For since

$$x^4 - 2ax^3 + (2a^2 - b^2)x^2 - 2a^3x + a^4 = 0;$$

$$\therefore\ x^4 - 2ax^3 + \quad 2a^2x^2 \quad - 2a^3x + a^4 = b^2x^2.$$

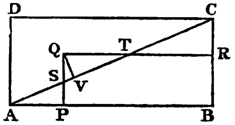
Adding a^2x^2 to both sides, and taking the square root, gives

$$x^2 - ax + a^2 = \pm \sqrt{(a^2 + b^2)}x,$$

or $x^2 - \{a \pm \sqrt{(a^2 + b^2)}\}x = - a^2$, the same quadratic as above obtained.

PROBLEM XII.

A gentleman has a garden 100 *feet long and* 80 *feet broad, and a walk is made of an equal width half round it, so as just to occupy half the garden; find the breadth of the walk.*



Let ABCD be the garden whose diagonal is AC, and PDR the walk. Let

AB $= a$, BC $= b$, AC $= c$, PB $= x$, BR $= y$, and $a + b + c = 2s$; then by hypothesis,

$$2xy = ab \text{ ...... } (1).$$

Now, since AP = CR, we have AB − BP = CB − BR, or BP − BR = AB − CB,

$$\text{hence, } x - y = a - b \text{ ......} (2).$$

Add twice (1) to the square of (2) and extract the root; then

$$x + y = \sqrt{a^2 + b^2} = c \text{ ...... } (3).$$

From (2) and (3) we obtain at once

$$x = \frac{a - b + c}{2} = s - b = \text{PB};$$

$$y = \frac{-a + b + c}{2} = s - a = \text{BR}.$$

Also, $a - x = \dfrac{a + b - c}{2} = s - c =$ AP, the width of the walk.

If $a = 100$, and $b = 80$, then $c = \sqrt{(a^2 + b^2)} = 20\sqrt{41}$, $s = \dfrac{a + b + c}{2}$ $= 90 + 10\sqrt{41}$, and AP $= s - c = 90 - 10\sqrt{41} = 25{\cdot}96876$ feet, the breadth of the walk.

Cor. If QV be drawn perpendicular to AC, then will QV = AP = CR.

For the triangles APS, QSV and ABC are equiangular; therefore (Euc. VI. 4.)

AB : BC :: AP : PS, or $a : b :: \dfrac{a + b - c}{2} : \text{PS} = \dfrac{ab + b^2 - bc}{2a}$;

$$\therefore \text{QS} = \text{BR} - \text{PS} = \frac{-a + b + c}{2} - \frac{ab + b^2 - bc}{2a} = \frac{ac + bc - (a^2 + b^2)}{2a}$$

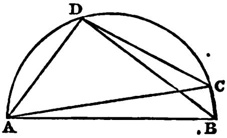
$$= \frac{ac + bc - c^2}{2a} = \frac{c}{a} \cdot \frac{a + b - c}{2} = \frac{c}{a}(s - c).$$

Again, AC : AB :: SQ : QV, or $c : a :: \dfrac{c}{a}(s - c) : \text{QV} = s - c$;

hence it is manifest that QV = AP = CR, and consequently Q is the centre of the circle inscribed in the triangle ADC.

PROBLEM XIII.

The lengths of three successive chords completing a semicircle are a, b, c; find the diameter of the circle.



Let AD $= a$, DC $= b$, CB $= c$, and AB $= x$, AC $= y$, BD $= z$.

Then (Euc. III. 31 and VI. D.)

$$\text{AB}^2 = \text{AD}^2 + \text{DB}^2;\ \text{or, } x^2 = a^2 + z^2\ldots\ldots(1)$$

$$\text{AB}^2 = \text{AC}^2 + \text{CB}^2;\ \text{or, } x^2 = c^2 + y^2\ldots\ldots(2)$$

$$\text{AB.CD} + \text{AD.BC} = \text{AC . BD};\quad \text{or, } yz = ac + bx\ldots\ldots(3).$$

From (1) and (2) we have $y^2 = x^2 - c^2$, and $z^2 = x^2 - a^2$; hence

$$y^2z^2 = (x^2-a^2)(x^2-c^2) = x^4 - (a^2 + c^2)\,x^2 + a^2c^2\ldots(4)$$

Equating the square of (3) to (4) we get

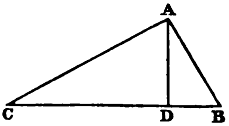
$$x^4 - (a^2 + c^2)\,x^2 + a^2c^2 = a^2c^2 + 2abcx + b^2x^2,$$

$$\text{or } x^3 - (a^2 + b^2 + c^2)\,x = 2abc\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

From which one value of x can be obtained when numerical values are given to the quantities a, b, c.

PROBLEM XIV.

Given the three sides of a plane triangle, to find its area.



Let ABC be a plane triangle, and denote the sides respectively opposite to the angles A, B, C by a, b, c. Draw AD perpendicular to BC, and let CD $= x$, and BD $= y$; then since

$$\text{AC}^2 - \text{CD}^2 = \text{AB}^2 - \text{BD}^2;$$

for each member is equal to the square of AD, we have the two equations

$$x + y = a\ldots\ldots(1).\qquad x^2 - y^2 = b^2 - c^2\ldots\ldots(2).$$

Dividing (2) by (1) gives

$$x-y=\frac{b^2-c^2}{a}\ \ldots\ldots(3)$$

Adding (1) and (3), and halving the sum, gives

$$x=\frac{a}{2}+\frac{b^2-c^2}{2a}=\frac{a^2+b^2-c^2}{2a}=\text{CD}.$$

Hence, $\text{AD}^2=\text{AC}^2-\text{CD}^2=b^2-x^2=(b+x)(b-x)$

$$=\left(b+\frac{a^2+b^2-c^2}{2a}\right)\left(b-\frac{a^2+b^2-c^2}{2a}\right)$$

$$=\frac{(a+b)^2-c^2}{2a}\cdot\frac{c^2-(a-b)^2}{2a};$$

$$\therefore\ 4a^2\,.\,\text{AD}^2=\{(a+b)^2-c^2\}\,\{c^2-(a-b)^2\}$$

$$=(a+b+c)\,(a+b-c)\,(c+a-b)\,(c-a+b).$$

But the area of a triangle is half that of a rectangle of the same base, and between the same parallels (Euc. I. 41); hence if Δ denote the area of the triangle ABC, we have ($\frac{1}{2}a\,.\,\text{AD}$ being equal to the area)

$$16\Delta^2=(a+b+c)\,(-a+b+c)\,(a-b+c)\,(a+b-c),$$

$$\text{or } \Delta^2=\frac{a+b+c)}{2}\cdot\frac{-a+b+c}{2}\cdot\frac{a-b+c}{2}\cdot\frac{a+b-c}{2}.$$

Let $s=\frac{a+b+c}{2}$; then $s-a=\frac{-a+b+c}{2}$, $s-b=\frac{a-b+c}{2}$

$s-c=\frac{a+b-c}{2}$, and the last equation becomes

$$\Delta^2=s\,(s-a)\,(s-b)\,(s-c);$$

$$\therefore\ \Delta=\sqrt{s(s-a)(s-b)\,(s-c)},$$

from which the following *practical rule* for determining the area of a triangle is deduced:

Find half the sum of the three sides of the triangle; from this half sum subtract each side separately; multiply together the half sum, and the three remainders, then the square root of the product will express the area of the triangle.

PROBLEM XV.

The extremities of two straight portions of a railway are to be connected by two circular curve lines; to determine the radius of one of the curves, when the radius of the other is given.

Let AB and CD be the two straight portions of the railway, and let it be required to connect the extremities of these, viz. B and C, by the two circular curves BS and SC.

Fig. 1.

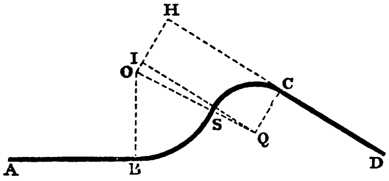
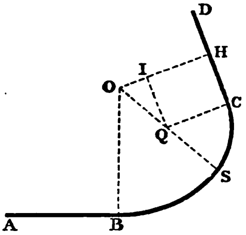


Fig. 2.



(1.) When the two lines of rail AB and CD form a *small* angle with each other, as in Fig. 1., then B and C must be connected by a curve of a serpentine form, which is technically called the *ogee* or S curve.

Let O be the centre of the circular portion BS, and Q that of the circular portion SC; then if OB and QC be drawn they must be perpendicular to AB and CD respectively, since the straight portions of the rail must be tangents to these circular curves, and the line OQ, joining the centres of the circular curves must pass through S their point of contact.

Produce DC and draw OH perpendicular to DC, and QI perpendicular to OH; then QH is a rectangle. Let OB $= r$, OH $= a$, CH $= b$, and QC $=$ QS $= x$; then we have OQ $= r + x$, QI $=$ CH $= b$, and OI $=$ OH $-$ IH $=$ OH $-$ QC $= a - x$, and therefore (Euc. I. 47) $\text{QO}^2 = \text{QI}^2 + \text{IO}^2$; that is

$$(r+x)^2 = b^2 + (a-x)^2, \text{ or } r^2 + 2rx + x^2 = b^2 + a^2 - 2ax + x^2;$$

$$\therefore\ 2(a+r)x = a^2 + b^2 - r^2 \text{ or } x = \frac{a^2 + b^2 - r^2}{2(a+r)} = \text{QC} \ldots . (1)$$

(2.) When the two lines of rail form with each other an angle not differing greatly from a right angle, then the connecting curve must be concave towards the same side as in Fig. 2. In this case we have OQ $= r - x$, and therefore

$$(r - x)^2 = b^2 + (a - x)^2 \text{ or } r^2 - 2rx + x^2 = b^2 + a^2 - 2ax + x^2;$$

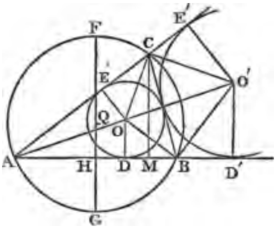
$$\text{whence } 2(r - a)x = a^2 + b^2 - r^2, \text{ or } x = \frac{a^2 + b^2 - r^2}{2(r-a)} \quad \ldots\ldots \quad (2)$$

Scholium. A straight line of railway cannot always be made, owing to unavoidable obstructions, and the expediency of making the line approach to densely populated localities; hence the necessity of railway curves, which are uniformly arcs of circles, whose radii are not allowed, by Act of Parliament, to be less than 80 chains, or 1 mile.

This useful problem has been given by Dr Rutherford for the purpose of enabling Railway Engineers to adapt their curves most advantageously to one another for safety.

PROBLEM XVI.

Given the three sides of a plane triangle, to determine the radii of the inscribed, escribed, and circumscribed circles.



Let ABC be the triangle, a, b, c the sides opposite to the angles A, B, C respectively, O the centre of the inscribed circle, OD and OE radii perpendicular to AB and AC. Draw AO, BO, CO, and let $\Delta =$ the area of the triangle ABC, and $r =$ OD or OE, the radius of the inscribed circle; then (Prob. XIV.) $\Delta^2 = s(s-a)(s-b)(s-c)$, and (*Euc.* I. 41, and II. *Def.* 1.) the triangle AOB $= \frac{1}{2}cr$. Similarly AOC $= \frac{1}{2}br$, and BOC $= \frac{1}{2}ar$; therefore, since the triangle ABC is composed of the three triangles AOB, AOC, BOC, we have

$$\Delta = \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr = r.\frac{a + b + c}{2} = rs;$$

$$\text{hence } r = \frac{\Delta}{s} = \frac{\sqrt{\{s(s-a)(s-b)(s-c)\}}}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \ldots (1).$$

Similarly, if O′ be the centre, and r_1 the radius of the escribed circle touching BC *exteriorly;* then ABC = AO′B + AO′C − BO′C = $\frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1$,

$$\therefore \quad \Delta = r_1 . \frac{b+c-a}{2} = r_1(s-a); \text{ therefore}$$

$$r_1 = \frac{\Delta}{s-a} = \sqrt{\frac{s(s-b)(s-c)}{s-a}} \quad \ldots\ldots\ldots\ldots \quad (2).$$

Also if r_2 and r_3 be the radii of the escribed circles touching AC and AB exteriorly, then

$$r_2 = \sqrt{\frac{s(s-a)(s-c)}{s-b}} \ldots\ldots\ldots (3); \quad r_3 = \sqrt{\frac{s(s-a)(s-b)}{s-c}} \ldots\ldots\ldots (4).$$

Taking the product of these radii we get

$$r\,r_1 r_2 r_3 = \sqrt{\frac{s^3(s-a)^3(s-b)^3(s-c)^3}{s(s-a)(s-b)(s-c)}} = s(s-a)(s-b)(s-c) = \Delta^2 \ldots (5).$$

Taking the reciprocals of r, r_1, r_2, r_3 we get

$$\frac{1}{r} = \frac{s}{\Delta}, \; \frac{1}{r_1} = \frac{s-a}{\Delta}, \; \frac{1}{r_2} = \frac{s-b}{\Delta}, \; \frac{1}{r_3} = \frac{s-c}{\Delta};$$

$$\text{but } \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta} = \frac{3s-(a+b+c)}{\Delta} = \frac{s}{\Delta};$$

$$\therefore \; \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \quad \ldots\ldots \quad (6).$$

Let Q be the centre, and R the radius of the circumscribed circle. Draw FQG perpendicular to AB bisecting it in H, and draw also the perpendicular CM; then since $\Delta = \frac{1}{2}$AB . CM, therefore CM $= \frac{2\Delta}{c}$. But (Euc. VI. C.) AC . CB = FG . CM, or $ab = 2\text{R} . \frac{2\Delta}{c}$; hence $4\text{R}\Delta = abc$, and consequently

$$\text{R} = \frac{abc}{4\Delta} = \frac{abc}{4\sqrt{\{s(s-a)(s-b)(s-c)\}}} \quad \ldots \quad (7).$$

Since $2\text{R} = \frac{abc}{2\Delta}$ and $r = \frac{\Delta}{s}$; therefore we have

$$2\text{R}r = \frac{abc}{2\Delta} \times \frac{\Delta}{s} = \frac{abc}{2s} = \frac{abc}{a+b+c} \quad \ldots \quad (8).$$

MISCELLANEOUS EXERCISES.

1. Determine the three factors of the expression $a^3 - 2abc - ab^2 - c^2a$, and find its numerical value when $a = 5$, $b = 3$, and $c = 1$.

Ans. $a(a+b+c)(a-b-c) = 45$.

2. Simplify the expression $\left\{\dfrac{x^6y^2 - 2x^4y^4 + x^2y^6}{(x-y)^6 + 8xy(x-y)^4 + 16x^2y^2(x-y)^2}\right\}^{\frac{1}{2}}$.

Ans. $\dfrac{xy}{x+y}$.

3. Find by a direct process the values of p and q, such that the expression $4x^4 - 12x^3 + px^2 + qx + 16$ may be a complete square.

Ans. $p = 25$, $q = -24$.

4. From what considerations is the meaning attached to $\frac{1}{0}$ deduced?

5. Determine the relative magnitudes of $\sqrt{10}$, $\sqrt[3]{30}$, and $\sqrt[4]{95}$, without extracting the specified roots of the numbers.

Ans. $\sqrt{10} > \sqrt[4]{95} > \sqrt[3]{30}$.

6. A person being asked what o'clock it was, said it is between eight and nine, and the hour and minute hands are exactly together; what was the time? Ans. 8 hrs. 43 min. $38\frac{2}{11}$ sec.

7. Find the values of x in the following equations.

(1) $\dfrac{x+7}{x-7} - \dfrac{x-7}{x+7} = \dfrac{7x}{x^2-73}$. Ans. $x = \pm 9$.

(2) $\sqrt{(a+\sqrt{x})} + \sqrt{(a-\sqrt{x})} = \dfrac{6a}{5\sqrt{(a+\sqrt{x})}}$.

Ans. $x = \dfrac{16a^2}{25}$ or $\dfrac{9a^2}{25}$

8. What number, consisting of two places of figures, is equal to the difference of the squares of its digits, and if 36 be added to it, the digits will be inverted? Ans. 48.

9. The pay of a Second Lieutenant for x days, at 5s. 6d. per diem, together with the pay of a First Lieutenant for y days at 7s. per diem, amounted to L.34, 4s. Now, if the Second Lieutenant is promoted to the rank of First Lieutenant, and the First Lieutenant to the rank of Second Captain, whose pay is 11s. per diem, their pay together for x and y days respectively would be increased by L.15, 12s.: find the values of x and y.

Ans. $x = 48$ days, $y = 60$ days.

10. Find the values of x and y from the simultaneous equations.

$$\sqrt{(6\sqrt{x} + 6\sqrt{y})} + \tfrac{1}{2}\sqrt{x} = 9 - \tfrac{1}{2}\sqrt{y}, \text{ and } x - y = 12.$$

Ans. $x = 16$, $y = 4$.

11. From two places, distant from each other 320 miles, two persons, A and B, set out at the same time to meet each other. A travelled 8 miles a day more than B, and the number of days in which they met was equal to half the number of miles B went in a day; how many miles did each travel per day? Ans. A 24 and B 16 miles.

12. What two numbers are those, whose difference, sum, and product, are to each other as the numbers 2, 3, and 5 respectively?

Ans. 2 and 10.

13. A piece of ground is to be enclosed with palisades, and it was found that if they were placed one foot apart, there would be 150 palisades wanting, but if placed a yard apart, there would be 70 remaining; find the number of palisades. Ans. 180.

14. After a certain number of men had been employed on a piece of work for 24 days, and had completed one-half of it, 16 men more were set on, and then the remaining half was finished in 16 days; how many men were employed at first, and what was the whole expense at 1s. 6d. a day per man? Ans. 32 men, and whole expense L.115, 4s.

15. The circumference of the fore-wheel of a carriage is c feet, and that of the hind-wheel is c' feet; in what distance will the fore-wheel make r revolutions more than the hind-wheel?

Ans. $\frac{cc'r}{c'-c}$.

16. It is required to find two numbers such, that if the square of the first be added to the second, the sum shall be 62, and if the square of the second be added to the first, it shall be 176. Ans. 7 and 13.

17. The fore-wheel of a carriage makes six revolutions more than the hind-wheel, in going 120 yards; but if the circumference of each wheel was increased by three feet, it would make only four revolutions more than the hind-wheel in the same space; what is the circumference of each wheel? Ans. 12 and 15 feet.

18. A sum of money is to be divided equally among a certain number of persons; now, if there had been 3 claimants less, each would have had L.150 more, and if there had been 6 more, each would have had L.120 less; required the number of persons, and the sum divided.

Ans. 9 persons, sum L.2700.

19. From each of sixteen foreign pieces of gold, of the same denomination, a person filed a fifth of its value, and then offered them all in payment at their nominal currency; but the fraud being detected, and the pieces weighed, they were found to be worth no more than L.11, 4s.; what was the original value of each piece? Ans. 17s. 6d.

20. A composition of tin and copper, containing 100 cubic inches, was found to weigh 505 ounces; how many ounces of each did it contain, supposing the weight of a cubic inch of copper to be 5¼ ounces, and that of a cubic inch of tin 4¼ ounces?

Ans. 420 oz. of copper, and 85 oz. of tin.

21. Given $xy(x^2+y^2)=3$ and $x^2y^2(x^4+y^4)=7$, to find the values of x and y.

$$\text{Ans. } x=\frac{\sqrt{5}+1}{2},\ y=\frac{\sqrt{5}-1}{2}$$

22. Given $xyz=231$, $xyw=420$, $xzw=660$, and $yzw=1540$, to find the values of x, y, z, w.

Ans. $x=3$, $y=7$, $z=11$, and $w=20$.

23. The sum of two numbers is 2, and the sum of their ninth powers is 32; find the numbers by a quadratic equation.

$$\text{Ans. } 1+\sqrt{\left\{\frac{-11+2\sqrt{34}}{3}\right\}},\ \text{and } 1-\sqrt{\left\{\frac{-11+2\sqrt{34}}{3}\right\}}.$$

24. Find the values of x and y from the equations

$$\frac{x}{y}=\frac{xy-ac}{bc-xy}, \text{ and } \frac{xy}{bc}=\frac{ax+ay-ac}{ay+bx-xy}.$$

Ans. $x=\pm(\pm ac)^{\frac{1}{2}}, y=\pm(\pm bc)^{\frac{1}{2}}$.

25. Extract the square root of $8\sqrt{-1}$ in the form of a binomial surd, and find the value of $(4+6\sqrt{-5})^{\frac{1}{2}}+(4-6\sqrt{-5})^{\frac{1}{2}}$.

Ans. $2+2\sqrt{-1}$, and 6.

26. Given $x+y+z=23$, $xy+yz+zx=167$ and $xyz=385$, to find x, y, z. Ans. $x=5, y=7, z=11$.

27. Given $x^2+xy=108$, $y^2+yz=69$, and $z^2+xz=580$, to find x, y, z. Ans. $x=9, y=3, z=20$.

28. If x varies as $\frac{z}{y^2}$, and z varies as $\frac{1}{x^2}$, then will x^3 vary as $\frac{1}{y^2}$.

29. If y is equal to the sum of three quantities, the first of which varies as x^2, the second as x, and the third is constant; find the equation expressing the relation between y and x, if when $x=1, 2, 3$, the corresponding values of y are 15, 37, and 69. Ans. $y=5x^2+7x+3$.

30. If A commence trade with L.5500, and if he add to this capital L.100 the first year, L.200 the second, L.300 the third, and so on; and if at the same time B commence trade with L.1000, and increase it in the first year by L.200, in the second by L.400, in the third by L.600, and so on; in how many years will these tradesmen have equal capitals, and what will be its amount? Ans. In 9 years—amount L.10,000.

31. The sum of three numbers in geometrical progression is 21, and the sum of their reciprocals is $\frac{7}{12}$; find the numbers.

Ans. 3, 6, 12.

32. If a is the mth term, and b the nth term of an harmonical progression, what is the $(m+n)$th term?

Ans. $\frac{(m-n)ab}{mb-na}$.

33. From a square pile of shot which had originally 24 courses, 14 entire courses and part of the 15th have been removed; and the number of shot left in the incomplete pile is known to be 3762; how many of the 15th course have been removed? Ans. 123 shot.

34. Expand $\frac{3x}{(3x+2)^2}$ in a series, both by the method of indeterminate coefficients, and the binomial theorem.

Ans. $\frac{1}{3x} - \frac{2.2}{(3x)^2} + \frac{3.2^2}{(3x)^3} - \frac{4.2^3}{(3x)^4} + \frac{5.2^4}{(3x)^5} - etc.$

35. Decompose $\frac{6x^2 - 8x + 12}{x^3 - 13x + 12}$ into three partial fractions.

Ans. $\frac{3}{x-3} - \frac{1}{x-1} + \frac{4}{x+4}$.

36. Find the term involving x^6 in the expansion of $(1+3x+5x^2+7x^3+ etc.)^5$. Ans. $17765x^6$.

37. Find the logarithms of ·875 and 3430 in terms of log 2 and log 7.

Ans. log 7 — 3 log 2 and 1 + 3 log 7.

38. Find all the roots of the equation $x^4+x^3-15x^2-19x-3=0$.

Ans. $x = 3{\cdot}99391857$, $x = -1{\cdot}09041796$

$x = -{\cdot}18526445$, $x = -3{\cdot}71823617$.

39. Given $256x - 87y = 1$, to find the least possible values of x and y in whole numbers. Ans. $x = 52$, $y = 153$.

40. In how many different ways is it possible to pay L.100 with seven-shilling pieces, and dollars of 4s. 6d. each?

Ans. In 31 different ways.

41. A farmer has two fields, for which he pays a rent of L.28; the rent of one field is L.4, 6s. 8d. per acre, and that of the other L.2, 10s.; how many acres are in each field?

Ans. 3 acres in the former, and 6 in the latter.

42. It is required to find two different isosceles triangles, such that their perimeters and areas shall be both expressed by the same numbers.

Ans. Sides of the one 29, 29, 40; and of the other 37, 37, 24.

43. It is required to find the sides of three right-angled triangles, in whole numbers, such that their areas shall be all equal to each other.

Ans. 58, 40, 42; 74, 24, 70; 113, 15, 112.

44. To find a whole number, such that if unity be added to it, and also to its half, the sums shall be squares. Ans. 48 or 1680.

45. Required the two least nonquadrate numbers x and y, such that the sum of their squares, and the sum of their cubes, shall be both squares. Ans. $x = 364$, and $y = 273$.

46. It is required to find two whole numbers, such that their sum shall be a cube, and their product and quotient squares.
Ans. 25 and 100, or 100 and 900, &c.

47. It is required to find three biquadrate numbers, such that their sum shall be a square. Ans. 12^4, 15^4, 20^4.

48. It is required to find three numbers in continued geometrical progression, such that their three differences shall be all squares.
Ans. 567, 1008, and 1792.

49. It is required to find three whole numbers, such that the sum or difference of any two of them shall be square numbers.
Ans. 856350, 949986, and 993250.

50. It is required to find four whole numbers, such that the difference of every two of them shall be a square number.
Ans. 1873432, 2288168, 2399057, and 6560657.

51. Given $x^{\frac{1}{x}}$ = 1·2655, to find a near approximate value of x.
Ans. 1·38736.

52. Given $x^y = 5000$, and $y^x = 3000$, to find the values of x and y.
Ans. x = 4·691445, and y = 5·510132.

53. How many figures would be required to express the 25th term of the series 2^1, 2^2, 2^4, 2^8, 2^{16}, *etc.*? Ans. 5050446 figures.

54. How may the series of weights 1, 3, 3^2, 3^3, *etc.* be employed to weigh 224 lbs.? Ans. $224 = 3^5 + 3^2 - (3^3 + 1)$.

55. Shew that if from the cube of any even number, four times the number itself be subtracted, the remainder will be divisible by 48.

56. There is a certain number of quantities of which the variations taken 8 together $= 80$, and taken 10 together $= 960$; how many must be taken away from the original number that, of the remaining quantities taken 2 together, the number of combinations may be 15? Ans. 6.

57. The amount of an annuity of P pounds to continue n years is $= a$ pounds, and the amount of the same annuity to continue $2n$ years is $= b$ pounds; what is the rate per cent. at compound interest?

$$\text{Ans. } 100\left\{\left(\frac{b-a}{a}\right)^{\frac{1}{n}}-1\right\}.$$

58. If the straight portion AB of a railway be produced 7 miles 250 yards, and the straight portion DC of another railway be produced 5 miles 1050 yards, they would meet and form a right angle. The railways are to be connected by two circular railway curves BS and SC, and the radius of the former is 8 miles 200 yards; find the radius of the other circular curve SC. Ans. 4 miles 763 yards $2\frac{1}{18}$ feet.

59. On the 19th of June 1846, a series of observations was made to determine accurately the time at Edinburgh, in latitude 55° 56′ 58″ N., longitude in time $12^{\text{m}}\,43_{\text{s}}{\cdot}5$ W. The mean of the times, by chronometer, at which the observations were made was $6^{\text{h}}\,4^{\text{m}}\,26_{\text{s}}{\cdot}83$; required, by interpolation, the sun's true declination from the Nautical Almanac, so as to obtain thereby the time with the utmost precision from the true zenith distance 71° 14′ 25″·3.

NOTE.—The column in page 1 of each month in the Nautical Almanac, titled "Diff. for 1 hour," is deficient in the requisite accuracy, when used as generally directed, near the solstice.

			d_1	d_2
1846. June 18.	Sun's dec. N.	23° 25′ 5·7″		
			+ 1′ 10·7″	
... 19.	...	23 26 16·4		− 24·8″
			+ 0 45·9	
... 20.	...	23 27 2·3		− 24·8
			+ 0 21·1	
... 21.	...	23 27 23·4		

Now 6h 4m 26·83s + 12m 43·50s = 6h 17m 10·33s = $\overset{\text{day}}{0{\cdot}262}$, nearly.

Whence by formula (2) page 261, we have $n = 2{\cdot}262$, $n-1 = 1{\cdot}262$ $n-2 = 0{\cdot}262$; $d_1 = 1'\,10''{\cdot}7$ or $70''{\cdot}7$, and $d_2 = -24''{\cdot}8$, hence

$$\text{Dec.} = 23^{\circ}\,25'\,5''{\cdot}7 + 1{\cdot}262 \times 70''{\cdot}7 - \frac{1{\cdot}262 \times 0{\cdot}262 \times 24''{\cdot}8}{1{\cdot}2}$$

$$= 23^{\circ}\,25'\,5''{\cdot}7 + 89''{\cdot}22 - 4''{\cdot}16 = 23^{\circ}\,26'\,30''{\cdot}76 \text{ N.}$$

The equation of time $= +\ 56^{\text{s}}{\cdot}57$ by the Nautical Almanac.

Hence the error of the chronometer was $4^{\text{s}}{\cdot}20$ slow.*

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* This refers to real observations, and may or may not be followed beyond the interpolation, at the pleasure of the instructor or student. The data required from the Nautical Almanac, and a few steps of the calculation, have been given to indicate the mode of procedure.

60. Required the moon's true longitude at the same place and time of day in July 1846. Here $t = 6^{\text{h}}\ 17^{\text{m}}\ 10^{\text{s}}\cdot33 = 0\cdot52385$ of 12^{h}, the interval in the Nautical Almanac, since the moon's longitude is given in it to every 12^{h}.

	h	° ′ ″	d_1	d_2	d_3	d_4
Long. on 18th at	0	= 60° 54′ 20″·1				
			+ 6° 16′ 50″·2			
	12	67 11 10·3		− 3′ 0″·0		
			+ 6 13 50·2		+ 19″·6	
19th	0	73 25 0·5		− 2 40·4		− 2″·1
			+ 6 11 9·8		+ 17·5	
	12	79 36 10·3		− 2 22·9		− 2·7
			+ 6 8 46·9		+ 14·8	
20th	0	85 44 57·2		− 2 8·1		
			+ 6 6 38·8			
	12	91 51 36·0			mean	− 2·4

$$\text{Hence}\quad \begin{array}{llll} d_1 = 6^\circ\ 16'\ 50''\cdot2 = 22610''\cdot2 & & n & = 3\cdot52385 \\ d_2 = -\ 3\ \ 0\cdot0 = -180\cdot0 & & n-1 & = 2\cdot52385 \\ d_3 = +\ \ 19\cdot6 = +\ 19\cdot6 & & n-2 & = 1\cdot52385 \\ d_4 = -\ \ 2\cdot4 = -\ 2\cdot4 & & n-3 & = 0\cdot52385 \\ & & n-4 & = -0\cdot47615 \end{array}$$

$$\text{Hence, Long.} = 60^\circ\ 54'\ 20''\cdot1 + 15^\circ\ 51'\ 4''\cdot75 - 5'\ 46''\cdot14 + 6''\cdot59$$

$$= 76^\circ\ 39'\ 45''\cdot3$$

The effects of d_4 being only $+\ 0''\cdot096$ are nearly insensible, and may for that reason be omitted.

61. The base of a triangle is b, and the perpendicular is p; required the side of a square inscribed in it?

Ans. $\dfrac{bp}{b+p}$.

62. Given the two sides and the line bisecting the vertical angle of a triangle, to find the base. Let a and b be the two sides, c the line bisecting the vertical angle, and x the base; then

$$x = (a+b)\left\{\frac{ab-c^2}{ab}\right\}^{\frac{1}{2}}.$$

63. Two trees standing on a horizontal plane are 120 feet asunder; the height of the higher is 100 feet, and that of the lower 80; whereabout in the plane must a person place himself, so that his distance from the top of either of the trees shall be equal to the distance between them?

Ans. $20\sqrt{21}$ feet from the bottom of the lower,
and $40\sqrt{3}$ feet from the bottom of the higher.

64. A town A was 30 miles from B, B 25 miles from C, and C 20 miles from A; whereabouts must a person place himself so as to be at equal distances from each? Ans. 15·118579 miles from each.

65. To find the side of a square, inscribed in a given semicircle, whose diameter is d. Ans. $\frac{1}{5}d\sqrt{5}$.

66. To find the side of an equilateral triangle, inscribed in a circle whose diameter is d; and that of another circumscribed about the same circle. Ans. $\frac{1}{2}d\sqrt{3}$, and $d\sqrt{3}$.

67. To find the side of a regular pentagon, inscribed in a circle whose diameter is d. Ans. $\frac{1}{4}d\sqrt{(10-2\sqrt{5})}$.

68. To find the sides of a rectangle, the perimeter of which shall be equal to that of a square, whose side is a, and its area half that of the square. Ans. $a+\frac{1}{2}a\sqrt{2}$ and $a-\frac{1}{2}a\sqrt{2}$.

69. Having given the side (10) of an equilateral triangle; to find the radii of its inscribed and circumscribed circles.

Ans. 2·88675 and 5·77350.

70. Having given the perimeter (12) of a rhombus, and the sum (8) of its two diagonals; to find the diagonals.

Ans. $4+\sqrt{2}$ and $4-\sqrt{2}$.

71. Having given the two contiguous sides (a, b) of a parallelogram, and one of its diagonals (d); to find the other diagonal.

Ans. $\sqrt{(2a^2+2b^2-d^2)}$.

72. Having given the perpendicular (300) of a plane triangle, the sum of the two sides (1155), and the difference of the segments of the base (495); to find the base and the sides.

Ans. 945, 375, and 780.

73. The lengths of the three lines drawn from the three angles of a plane triangle to the middle of the opposite sides, being 18, 24, and 30, respectively; it is required to find the sides.

Ans. 20, 28·844, and 34·176.

74. In a plane triangle, there is given the base (b), the area (a^2), and the difference of the sides (c); to find the sides and the perpendicular.

Ans. The perp. $p=\frac{2a^2}{b}$, and the sides $=$

$$b\sqrt{\left\{\frac{p^2}{b^2-c^2}+\frac{1}{4}\right\}}\pm\frac{c}{2}.$$

75. The length of two lines that bisect the acute angles of a right angled plane triangle being a and b, it is required to determine the triangle.

Ans. If $a = 40$ and $b = 50$, the sides are 35·80737, 47·40728, and 59·41143.

76. Having given the lengths (a, b) of two chords, cutting each other at right angles in a circle, and the distance (c) of their point of intersection from the centre; to determine the diameter of the circle.

Ans. $\sqrt{\{\frac{1}{2}(a^2 + b^2) + 2c^2\}}$.

77. Having given the sides of a trapezium, inscribed in a circle, equal to 6, 4, 5, and 3, respectively; to determine the diameter of the circle.

Ans. $\frac{1}{20}\sqrt{(130 \times 133)}$ or 6·57457.

78. In a plane triangle, having given the perpendicular (p), and the radii (r, R) of its inscribed and circumscribed circles; to determine the triangle.

$$\text{Ans. The base} = \frac{2r\sqrt{(2p\text{R} - 4r\text{R} - r^2)}}{p - 2r}.$$

79. Having given the base of a plane triangle equal to $2a$, the perpendicular equal to a, and the sum of the cubes of its other two sides equal to three times the cube of the base; to determine the sides.

Ans. $a(2 + \frac{1}{3}\sqrt{6})$ and $a(2 - \frac{1}{3}\sqrt{6})$.

80. Given the area (100) of an equilateral triangle ABC, whose base BC falls on the diameter, and vertex A in the middle of the arc of a semicircle; required the diameter of the semicircle. Ans. $20\sqrt[4]{3}$.

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